Mathematical Reviews

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FOUNDATIONS

Rougier, Louis. La relativité de la logique. J. Unified Sci. (Erkenntnis) 8, 193–217 (1939). [MF 2671]

✶Quine, Willard Van Orman. Mathematical Logic. W. W. Norton & Co., Inc., New York, 1940. xiii+348 pp. $4.00.

The scope of this work is indicated by its chapter titles: I. Statements; II. Quantification; III. Terms; IV. Extended theory of classes; V. Relations; VI. Number; VII. Syntax. The book presupposes no specialized training in logic or mathematics, although intended only for the serious and intelligent student. The truth-functional conditional (involving a binary connective related to "material implication") is here contrasted to logical implication which connects only names of statements, and is semantic. Extensive use of calculation by truth tables (exhibited or left to the reader) makes unnecessary ingenious constructive proofs incident to a purely postulational deductive treatment. Quantification is reduced to a single primitive form, the complete universal. An ϵ-relation of "membership" (here extended to apply to non-classes if any) provides a basis for the method of abstraction, and for description. The traditional logical antinomies are avoided by a method analogous to von Neumann's, rendering unnecessary a restrictive theory of types. Functions are defined in terms of relations, relations in terms of class theory. Chapter VI provides in brief compass a formal basis for arithmetic and indeed for the theory of functions of a real variable. The concluding chapter presents a metamathematical theory about such formalism as has been developed in the earlier chapters. Much informal descriptive comment, appropriately applied, serves to guide and warn the reader. The notation follows, in the main, that of Principia. The use of a modification of Sheffer's stroke-function, the systematic employment of "corners" when needed for naming, the regular adoption of the λ of abstraction (as by Church) and other special modifications are superficially conspicuous. The author's list of defined symbols is unexpectedly extensive for a work of this compass as is his list of stated theorems and metatheorems. Numerous appropriate remarks in fine print deal with the history of special symbols or methods. The author takes definite stand on a large number of controversial issues. Many students of the subject may find that their own views have been summarily rejected or merely ignored, and will find no reason to accept some of the judgments here unapologetically announced.

A. A. Bennett (Providence, R. I.).

Quine, W. V. and Goodman, Nelson. Elimination of extra-logical postulates. J. Symbolic Logic 5, 104–109 (1940).

The authors consider postulate systems obtained by adjoining to ordinary logic certain additional primitives, together with postulates concerning them, which postulates are not deducible on logical grounds. Their problem is that of eliminating these extra-logical postulates, that is, finding explicit definitions of these primitives in terms of other primitives, which in turn are not subject to any postulates, such that the original postulates become deducible from the definitions on purely logical grounds. They find that such an elimination is possible if, and only if, the original system has a logical model. Of course this presupposes that the underlying logic must have a certain degree of strength, in particular it must include the processes for manipulating descriptions; but this degree of strength does not exceed that possessed by several systems in common use. In conclusion the authors discuss the significance of their result. Of course it upsets many notions of postulational economy. The authors advance reasons to the effect that significant criteria of economy will exist only in systems which are relatively complete in a sense due to Tarski; this is stated to be a property which fails for many useful systems and is normally difficult to establish when it holds. For other sorts of systems "the extensive economies here shown to be possible will seldom be distinguishable from those effected in any of the usual ways."

H. B. Curry.

Pólya, G. Sur les types des propositions composées. J. Symbolic Logic 5, 98–103 (1940).

Jevons proposed the problem of finding the number of different types of Boolean functions of n variables (without parameters), two functions being of the same type if obtainable from one another by permuting variables, or replacing variables by their negatives, or both. This number is the same as the number of distinct ways of coloring with two colors the 2^n vertices of an n dimensional hypercube, counting colorings as distinct if not obtainable from each other by an operation of the group Σ (of order $n!2^n$) of rotations and symmetries of the hypercube. The problem is a special case of a class of enumeration problems associated with a group, already treated by the author [Acta Math. 68, 145–254 (1937)]. The number of types of function with s terms, $s=0, \cdots, 2^n$, is found to be the coefficient of x^s in a certain polynomial obtained by counting the number of permutations of each cyclic type in the representation of Σ as a permutation group on the vertices. The numbers given by the author for $n=4$ differ in three cases from those computed by W. K. Clifford, using another method.

O. Frink (State College, Pa.).

Wernick, William. Functional dependence in the calculus of propositions. Amer. Math. Monthly 47, 602–605 (1940). [MF 3260]

Two conditions are given which are necessary and sufficient that a two-valued function of several two-valued variables be independent of a particular variable. From these are derived conditions necessary and sufficient that such a function be dependent on all its variables. It is shown

that a function whose values are 0 or 1 is dependent on all its variables if the sum of all functional values is odd.

O. Frink (State College, Pa.).

McKinsey, J. C. C. Proof that there are infinitely many modalities in Lewis's system S_2. J. Symbolic Logic 5, 110–112 (1940).

An interpretation of Lewis' system of strict implication S_2 is given in which the elements consist of all sets of integers, positive, negative or zero. Logical disjunction, conjunction and negation are represented by set union, intersection and complement; and $\Diamond A$ (meaning A is possible) is interpreted as the union of A and the set of all successors of elements of A. It is shown that all the postulates and rules of procedure of S_2 hold, and that, in this interpretation, no two modalities are equivalent if they do not have the same number of diamonds. Hence no two of the infinitely many modalities $\Diamond p, \Diamond\Diamond p, \cdots, \Diamond_n p, \cdots$ are provably equivalent in S_2. W. T. Parry has shown [J. Symbolic Logic 4, 137–154 (1939); these Rev. 1, 131] that in S_3 and all stronger systems there are only a finite number of distinct modalities. It is interesting to note that Lewis' existence postulate, that there exist elements p and q such that $\sim(p \prec q)\cdot\sim(p \prec \sim q)$, also holds in this interpretation, if p is the universal set and q the set of all even integers.

O. Frink (State College, Pa.).

McKinsey, J. C. C. Postulates for the calculus of binary relations. J. Symbolic Logic 5, 85–97 (1940).

This paper presents, for the first time, a postulational formulation of the Peirce-Schröder theory of relations. The primitive ideas employed are a class K, a dyadic relation $\angle$ and a binary operation $|$. Thirty theorems, derived from the postulates, give us the basic properties of $\angle$ and $|$, and three meta-theorems inform us about the nature of the realizations of the postulates. The consistency and the independence of the postulates are established and the independence of the primitive ideas discussed. Besides the primitives K, $\angle$, $|$, the postulates also contain certain defined concepts. These are the Boolean elements 0 and 1, the notion of a Boolean "atom" (prime), and the notion of a "complete atomic Boolean algebra." A Boolean element p is an atom if the only values of x satisfying the relation $x\angle p$ are p and 0. A complete atomic Boolean algebra is one in which (1) every non-0 element contains an atom and (2) the "sum" and "product" of every set (finite or infinite) of elements exist, the sum and product being definable (in an obvious manner) in terms of $\angle$. Following are the postulates. (P_1) K is a complete atomic Boolean algebra with respect to $\angle$. (P_2) If x and y are in K, then $x|y$ is in K. (P_3) If x, y, z are in K, then $x|(y|z)=(x|y)|z$. (P_4) If x, y, u, v are in K and $x\angle y$ and $u\angle v$, then $(x|u)\angle y|v$. (P_5) If x is in K and $x\neq 0$, then $1|(x|1)=1$. (P_6) If x, y are in K and p is an atom of K such that $p\angle x|y$, then there exist atoms q, r in K such that $q\angle x$, $r\angle y$, $p\angle q|r$. (P_7) If p, q, r are atoms of K such that $p|q\neq 0$, $q|p\neq 0$, $p|r\neq 0$, $r|p\neq 0$, then $q=r$.

A "complete atomic relational realization" of the postulates is the following. Let K be the class of ordered couples of individuals from some domain of individuals; $x\angle y$ means "x is a subset of y"; $x|y$ means the "relative product" of x and y (the set of couples (α, β) such that there is a couple (α, γ) in x and a couple (γ, β) in y). The principal theorem of the paper is: Every two realizations of the postulates which possess the same number (finite or infinite) of homogeneous atoms, are isomorphic (an atom p is homogeneous if $p|p\neq 0$).

B. A. Bernstein (Berkeley, Calif.).

***Gödel, Kurt. The Consistency of the Continuum Hypothesis.** Annals of Mathematics Studies, no. 3. Princeton University Press, Princeton, N. J., 1940. 66 pp. $1.25.

[Notes by George W. Brown on lectures delivered at the Institute for Advanced Study during the autumn term 1938–1939.] Gödel shows that, if a certain axiomatic system Σ for set theory is consistent, the system is consistent which results from Σ by introducing as additional axioms the axiom of choice (in a strong form, which provides for the simultaneous choice, by a single relation, of an element from each non-vacuous set of the universe of sets) and Cantor's generalized continuum hypothesis ($2^{\aleph_\alpha}=\aleph_{\alpha+1}$ for any α). His method consists in constructing within Σ a model Δ of Σ, in which model the axiom of choice and the generalized continuum hypothesis hold.

In more detail, the method is this. In an axiomatic system, certain notions are treated as primitive and not defined, and every proposition about those notions must follow from only those properties of them which are expressed by the axioms. For Σ, there are three undefined notions, $\mathfrak{Cls}$ (class), $\mathfrak{M}$ (set) and ϵ (the membership relation). From the undefined notions, with the aid of the axioms and logical principles, other notions can be defined. In particular, three are defined designated as $\mathfrak{Cls}_l$, $\mathfrak{M}_l$ and ϵ_l. From the axioms of Σ a set of propositions are deduced which have exactly the forms of the several axioms of Σ, the axiom of choice, and the generalized continuum hypothesis, respectively, except that $\mathfrak{Cls}_l$, $\mathfrak{M}_l$ and ϵ_l occur throughout in place of $\mathfrak{Cls}$, $\mathfrak{M}$ and ϵ, respectively. This reduces the consistency of the augmented system to that of Σ. For if a contradiction were a consequence of the enlarged axiom list, that is, as axioms for $\mathfrak{Cls}$, $\mathfrak{M}$ and ϵ, then a contradiction would follow likewise in terms of $\mathfrak{Cls}_l$, $\mathfrak{M}_l$ and ϵ_l from the aforesaid set of propositions. Since those propositions are theorems of the system Σ, this contradiction would be a consequence of the axioms of Σ alone. In Zermelo's axiomatization of set theory, one of the axioms (the Aussonderungsaxiom) employed as an intuitive notion the notion of a "definite property" of sets. Those elements of a given set which have a given definite property were constituted a subset of the given set. The notion of "definite property" is lacking in clarity, and subsequent axiomatizations of set theory have eliminated this notion as an intuitive presupposition by stating explicitly how "definite properties" are to be constructed. The Skolem "paradox," and now these results of Gödel, depend on this.

Gödel's system Σ is essentially due to Bernays [J. Symbolic Logic 2, 65–77 (1937)], and is equivalent to von Neumann's system S^*+VI with Ax. III 3* replaced by Ax. III 3 [J. Reine Angew. Math. 160, 227–241 (1929)]. A property of a set is treated in extension, as the class of sets which have the property; similarly a property of n sets, that is, a relation among n sets, is treated as the class of the ordered n-tuples of sets which stand in the relation. This makes possible a very succinct delimitation of "definite property." The operations for constructing "definite properties" enter through axioms concerning the existence of classes (Axs. B1–B8). In the notation of propositional functions, they give R under the following eight circumstances, where A and B represent propositional functions already obtained: $R(x, y)\equiv x\epsilon y$; $R(x)\equiv A(x)$-and-$B(x)$; $R(x)\equiv$not-$A(x)$; $R(x)\equiv$exists-y-such-that-$A(y, x)$; $R(y, x)\equiv A(x)$;

$R(x,y)\equiv A(y,x)$; $R(x,y,z)\equiv A(y,z,x)$; $R(x,y,z)\equiv A(x,z,y)$. Sets are classes (Ax. A1). Only sets can occur as members of classes (Ax. A2). Classes with the same extension are identical (Ax. A3). Classes occur which are not sets, for example, the universal class V; and this is provable using an axiom which has the effect of excluding the existence of infinite sequences of sets such that each set of the sequence has the next as member (Ax. D). The definition of the ordered n-tuples used in treating relations is based on an axiom by which an unordered pair of sets is a set (Ax. A4). The other axioms of Σ are the axiom of infinity (Ax. C1), axioms asserting the existence of a set including the sum of the elements of a given set (Ax. C2) and a set including the set of subsets of a given set (Ax. C3), and the axiom of substitution which takes the place of the Aussonderungs-axiom (Ax. C4).

Some space is devoted to a preliminary development of abstract set theory, partly from these axioms, and partly from these axioms and the axiom of choice. (*8.3 should read: $\overline{\overline{\alpha+1}}\leqq\overline{\overline{\alpha}}^2$ for $\alpha>1$.) The deductions are informal, but could be formalized in the calculus of propositional functions of first order. The operations for constructing sets and classes given by Axs. A4, B1–B8, Gödel writes, after certain modifications, as eight fundamental binary operations which, applied to a pair of sets, yield a set. The idea for the model Δ is to take as sets only those which can be generated by these operations and the operation, applied at certain stages of the generating process, of forming the set of all sets already obtained. This process of generation is expressed in the definition of a function F of an ordinal α by transfinite induction. To give scope to the nine operations, the ordinals are separated into congruence classes J_0–J_8 modulo 9; and the binary character of the fundamental operations necessitates a mapping of the pairs of ordinals on each of the congruence classes J_1–J_8. The values of F taken for α in J_i ($i>0$) are sets given by the fundamental operations from pairs of sets obtained as earlier values of F; and the values of F taken for α in J_0 are the sets of all values previously obtained. The sets taken as values of F are called "constructible sets" and constitute a class L. The "constructible classes" are the classes of which both the elements and the intersections with constructible sets are constructible sets. $\mathfrak{Cls}_l$ is the notion "constructible class"; $\mathfrak{M}_l$ is the notion "constructible set"; and ϵ_l is the ϵ-relation under restriction of the domains of the relation to the constructible classes and sets. It now turns out that all the axioms of Σ are satisfied for the model, that is, they are theorems of Σ when read with $\mathfrak{Cls}_l$, $\mathfrak{M}_l$ and ϵ_l in place of $\mathfrak{Cls}$, $\mathfrak{M}$ and ϵ. Furthermore the proposition that every set is constructible holds in the model, that is, $V_l=L_l$ is a theorem of Σ, where V_l and L_l are defined in terms of $\mathfrak{Cls}_l$, $\mathfrak{M}_l$ and ϵ_l in the same way as V and L in terms of $\mathfrak{Cls}$, $\mathfrak{M}$ and ϵ. This shows that the proposition $V=L$ is consistent with the axioms of Σ, if Σ itself is consistent.

The prospectus now calls for showing that the axiom of choice and the generalized continuum hypothesis, relativized for the model, that is, read with $\mathfrak{Cls}_l$, $\mathfrak{M}_l$ and ϵ_l in place of $\mathfrak{Cls}$, $\mathfrak{M}$ and ϵ, are theorems of Σ. In fact, these propositions can be deduced from the relativized axioms of Σ and the proposition $V_l=L_l$, already proved as theorems of Σ. In other words, dropping the subscript l throughout, the (unrelativized) axiom of choice and generalized continuum hypothesis become theorems of Σ when the proposition $V=L$, that is, every set is constructible, is added to Σ as a new axiom. For the axiom of choice, this is immediately evident, since each constructible set can be assigned as its "order" the least ordinal α for which it is taken as value of F. Then the element to be chosen from any non-vacuous set can be specified as the one having the least order.

To treat the generalized continuum hypothesis $2^{\aleph_\alpha}=\aleph_{\alpha+1}$ (the axiom of choice now being available), Gödel considers the set (written $F``\omega_\alpha$) of all constructible sets of orders less than ω_α (where ω_α is the first number of Cantor's $(2+\alpha)$th number class). $\aleph_\alpha$ is the cardinal of the set of all ordinals less than ω_α. The intersection of this set with the congruence class J_0 has the same cardinal, and on this subset all values taken by F are distinct. Hence $\overline{\overline{F``\omega_\alpha}}=\aleph_\alpha$ (where $\overline{\overline{x}}$ denotes the cardinal number of $x$). Therefore $\overline{\overline{\mathfrak{P}(F``\omega_\alpha)}}=2^{\aleph_\alpha}$ (where $\mathfrak{P}(x)$ denotes the set of subsets of x). By Cantor's theorem and the comparability theorem (based on the axiom of choice), $2^{\aleph_\alpha}\geqq\aleph_{\alpha+1}$. Now consider any member of $\mathfrak{P}(F``\omega_\alpha)$, that is, any subset of $F``\omega_\alpha$. By $V=L$, this subset must be taken as value of F for some ordinal δ. By a detailed examination of the definition of F, Gödel is able to show that the subset must then necessarily be taken as value of F for some ordinal less than $\omega_{\alpha+1}$. Hence $\mathfrak{P}(F``\omega_\alpha)$ is a subset of $F``\omega_{\alpha+1}$. Therefore $\overline{\overline{\mathfrak{P}(F``\omega_\alpha)}}\leqq\overline{\overline{F``\omega_{\alpha+1}}}=\aleph_{\alpha+1}$. Therefore $2^{\aleph_\alpha}=\aleph_{\alpha+1}$.

S. C. Kleene (Madison, Wis.).

TOPOLOGY

✶Tukey, John W. Convergence and Uniformity in Topology. Annals of Mathematics Studies, no. 2. Princeton University Press, Princeton, N. J., 1940. ix+90 pp. \$1.50.

The structure of a space is of interest in topology to the extent that it permits the formulation of limiting processes. In metric space the structure may be expressed in terms of closure, neighborhood and convergence of sequences with equivalent results. Each of these notions has been used as a basis of topology. The book under review has as one of its objects the statement of general conditions under which the equivalence of these notions is retained. Besides this local structure, which in metric space comes from a consideration of the spheres $S(x,e)$ as a function of $e>0$ for each x, there is the uniform structure of the space, arising from the $S(x,e)$ considered as a function of x for each $e>0$, in terms of which such notions as uniform continuity and Cauchy convergence are formulated. Lately this structure has been treated in general spaces. The author gives an account of some of these results and some new ones by methods of his own.

The theory of convergence, which derives from E. H. Moore and H. L. Smith, is based on functions $x(a)$ on a directed system to a set X; $A=\{a\}$ is a directed system if there is a transitive order $a'>a''$, defined for some pairs in A, such that for any a', a'' there is an $a>a'$, a''. The notion of convergence, $x(a)\to x$, is subject to a condition similar to the theorem that a subsequence of a convergent sequence converges to the limit of the original sequence. This is formulated as follows: if $B\subset A$ has the property that for each $a\epsilon A$ there is a $b\epsilon B$ such that $b>a$, then $x(a)\to x$ implies $x(b)\to x$. Closure in a set X is a linear set function on subsets

of X to subsets of X which preserves the null set. A neighborhood system in X satisfies the condition that the intersection of two neighborhoods of a point contains a neighborhood of the point. Starting with any one of these notions as primitive, each of the others is so defined that it satisfies the conditions stated above. The author defines a space as a set X in which these concepts are given. Special classes of spaces are treated separately: T spaces are those for which the closure $\bar{H} \supset H$ and $\bar{\bar{H}}=\bar{H}$; T_1 spaces are T spaces in which a point is a closed set; normal spaces are defined as usual.

Following this the author treats compactness. One of several equivalent definitions states that X is compact if every $x(a)$ on a directed system to X has a cluster point, that is, a point x such that for each neighborhood $N(x)$ and a' there is an $a>a'$ for which $x(a)\epsilon N(x)$. An interesting result here concerns an ultraphalanx. The author defines a phalanx as a function $x(\alpha)$ on the directed system whose elements α are the finite subsets of a given set and for which $\alpha'>\alpha''$ means $\alpha'\supset\alpha''$. An ultraphalanx is a phalanx $x(\alpha)$ such that for any $H\subset X$ either $x(\alpha)\epsilon H$ for $\alpha>\alpha'$ or $x(\alpha)\epsilon X-H$ for $\alpha>\alpha''$. The convergence of all ultraphalanxes in X is equivalent to the compactness of X. In T spaces compactness is equivalent to bicompactness. Next there is a discussion of normal spaces. This is based on a calculus of coverings $\mathfrak{M}$ of the space X. Coverings are ordered by writing $\mathfrak{M}<\mathfrak{N}$ if for each $M\epsilon\mathfrak{M}$ there is an $N\epsilon\mathfrak{N}$ such that $M\subset N$. From a covering $\mathfrak{M}$ two coverings are derived as follows. Let H be a subset of X and $S(H,\mathfrak{M})$ be the union of all $M\epsilon\mathfrak{M}$ which meet H. The class of $S(M,\mathfrak{M})$ for all $M\epsilon\mathfrak{M}$ is a covering $\mathfrak{M}^*$ and the class of $S(x,\mathfrak{M})$ for all $x\epsilon X$ is a covering $\mathfrak{M}^\Delta$. These coverings are related by $\mathfrak{M}^\Delta<\mathfrak{M}^*<\mathfrak{M}^{\Delta\Delta}$. The development of these ideas leads to a solution of the metrization problem and to the theorem on the existence of non-constant continuous functions in normal space.

Families of coverings made up of open sets and suitably restricted underlie the study of the uniform structure of a space. The author defines a uniformity in a space X as a family of open set coverings $\{\mathfrak{U}\}$ such that if $\mathfrak{U}<\mathfrak{U}'$ and $\mathfrak{U}\epsilon\{\mathfrak{U}\}$ then $\mathfrak{U}'\epsilon\{\mathfrak{U}\}$, and if $\mathfrak{U}',\mathfrak{U}''\epsilon\{\mathfrak{U}\}$ then there is a $\mathfrak{U}\epsilon\{\mathfrak{U}\}$ such that $\mathfrak{U}^*<\mathfrak{U}',\mathfrak{U}''$. If for each $x\epsilon X$ the open sets containing x are identical with the $S(x,\mathfrak{U})$ for all $\mathfrak{U}$ in some uniformity in X, the uniformity is said to agree with the topology of X. Such a space the author calls a struct. A mapping $y=f(x)$ on a struct to a struct is called uniformly continuous if for each covering $\mathfrak{U}$ of the image struct there is a covering $\mathfrak{B}$ of the original struct such that for all x, $f(S(x,\mathfrak{B}))\subset S(f(x),\mathfrak{U})$. It is shown that a continuous function on a compact struct is uniformly continuous. The concept of a Cauchy sequence is generalized by defining a Cauchy mapping as a mapping $x(a)$, on a directed system A into a struct, such that for any $\mathfrak{U}$ of the uniformity in the struct there are an $S(x',\mathfrak{U})$ and an $a'\epsilon A$ such that $x(a)\epsilon S(x',\mathfrak{U})$ if $a>a'$. If every phalanx which is a Cauchy mapping is convergent in a struct, the struct is called complete. It is shown that every struct may be uniformly imbedded in a complete struct. These ideas are related to the work of A. Weil and the reviewer.

The book concludes with a chapter on product spaces, an account of some interesting examples and a provocative discussion of such questions as: What is topology?; Which separation axioms are important?; and No transfinite numbers wanted. There is a bibliography and index. The collection of ideas in this book, only some of which have been discussed here, is interesting and the exposition gives a systematic account of many of the principal topics in topology. A few of the theorems stated without proof are inaccurate and one may wonder whether the new terms employed contribute clarity in proportion to their number. The text, which is lithoprinted, is somewhat marred by typographical slips. These are rather small objections to a work which will profitably engage the attention of all who are interested in contemporary developments in topology.

L. W. Cohen (Lexington, Ky.).

Krasner, M. Un type d'ensembles semi-ordonnés et ses rapports avec une hypothèse de M. A. Weil. Bull. Soc. Math. France **67**, 162–176 (1939). [MF 2586]

The author calls a set "semi-ordered" if its ordering relation satisfies: I. $a_1<a_2<\cdots<a_n$ implies $a_1<a_n$; II. for each a and b there is an element c such that (i) either $c>a$ or $c=a$, and (ii) either $c>b$ or $c=b$. A subset of a semi-ordered set is "confinal" if each element of the set is exceeded by some element of the subset. The author attempts a proof by transfinite induction that every semi-ordered set contains a confinal subset whose ordering relation may be weakened in such a way that the subset with its weaker ordering is isomorphic to the class of all finite subsets of a given set (where $>$ means proper inclusion). The additional hypothesis of transitivity is necessary. Consider a totally intransitive ($a>b$, $b>c$, $a>c$ implies either $a=b$ or $b=c$) semi-ordered set which contains no confinal finite subset; such a set may easily be constructed and clearly denies the conclusion of the theorem. In the transitive case the author's proof can be made simpler and more intuitive.

The interest in this result lay in a reduction of a hypothesis of André Weil concerning "projective systems" (a special case of "inverse mapping systems" of Steenrod [Amer. J. Math. **58**, 661–701 (1936)]). A footnote states that Aronszajn has (had since 1935) counter-examples to the hypothesis. Other examples, showing that the addition of topological conditions of reasonable strength do not validate the hypothesis, will be found in the reviewer's Princeton thesis [1939]. *J. W. Tukey* (Princeton, N. J.).

Cohen, L. W. On topological completeness. Bull. Amer. Math. Soc. **46**, 706–710 (1940). [MF 2664]

This paper is concerned with the notion of uniformity as introduced recently by Weil [Sur les espaces à structure uniforme et sur la topologie générale, Actual. Sci. Ind., Paris, 1938] and by the author [Duke Math. J. **3**, 610–617 (1937) and **5**, 174–183 (1939)]. The author says: "It is the object of this paper to show that Weil's space is a special case of the space $S_{\mathrm{I,II,III}}$ and that the notion of Cauchy family in this space leads to the same theory of completeness as previously developed." Previously the author has shown that any "uniform space" in his sense could be imbedded in one which was sequentially complete. It is now shown that Weil's notion is a special case of the author's notion. The possibility of the converse, in the sense that every "uniform space" in the author's sense can be mapped on a "uniform space" in Weil's sense by a transformation which is uniformly continuous in both directions, is not excluded. It is shown that the imbedding spaces constructed earlier are actually complete in Weil's sense. It is neither asserted nor is it true that every space sequentially complete in the author's sense is also complete in the sense of Weil.

J. W. Tukey (Princeton, N. J.).

Krishna Murti, S. B. A set of axioms for topological algebra. J. Indian Math. Soc. (N.S.) **4**, 116–119 (1940). [MF 3273]

The author gives a set of axioms for topology in terms of $AJB=A\bar{B}+B\bar{A}$. This operation is symmetric, idempotent, distributive, and satisfies $AJ0=0$ and $AJ(AJB)=AJB$. He proves his axioms are equivalent to Kuratowski's ["Topologie"], with $\bar{p}=p$ omitted. *G. Birkhoff.*

Monteiro, António. Caractérisation des espaces de Hausdorff au moyen de l'opération de dérivation. Portugaliae Math. **1**, 333–339 (1940). [MF 2893]

M. Fréchet's problem of characterizing Hausdorff space by means of an operation of derivation is solved. It is assumed, in addition to the usual axioms of derivation, that if distinct points a_1 and a_2 belong to A' then A consists of two disjoint non-empty parts A_1 and A_2 such that A_1' does not contain a_2 and A_2' does not contain a_1. Some of the auxiliary results are more general than necessary for their use here, but it is shown by example that the conditions under which these auxiliary results follow are indispensable. *J. F. Randolph* (Ithaca, N. Y.).

Monteiro, Antônio et Ribeiro, Hugo. Sur l'axiomatique des espaces (V). Portugaliae Math. **1**, 275–288 (1940). [MF 2227]

The primitive notions of derivative, closure, frontier, interior, exterior, edge and border for sets of an abstract space are used in turn to axiomatize the neighborhood space (V) of Fréchet. The precedence of work of other authors is recognized. *J. F. Randolph* (Ithaca, N. Y.).

Ribeiro, Hugo B. Sur l'axiomatique des espaces topologiques de M. Fréchet. Portugaliae Math. **1**, 259–274 (1940). [MF 2228]

This note is a record of a seminar conducted at Lisbon by Antonio Monteiro; the results and methods are similar to those of the paper reviewed above. *J. F. Randolph.*

Eidelheit, M. Eine Bemerkung über lineare topologische Räume. Revista Ci., Lima **42**, 475–477 (1940). [MF 2901]

An example of a linear Hausdorff space in which the operation of addition is sequentially continuous, but not continuous according to the customary neighborhood definition. *J. A. Clarkson* (Philadelphia, Pa.).

Proskuriakov, I. Über einige Eigenschaften der im kleinen kompakten und im kleinen bikompakten Räume. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 153–156 (1939). (Russian. German summary) [MF 2109]

Let R be a bicompact Hausdorff space (or a compact regular space satisfying the first axiom of countability). A subset A of R is locally bicompact (locally compact) if and only if $\bar{A}-A$ is closed in R. Further, if A is a dispersed subset of R, that is, if only the empty subset of A is dense-in-itself, then A is the difference between a bicompact Hausdorff (compact regular) space and its perfect kernel if and only if A is locally bicompact (locally compact). *H. Wallman* (Princeton, N. J.).

Proskuriakov, I. Endliche Mengensysteme in topologischen Räumen. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 141–151 (1939). (Russian. German summary) [MF 2108]

This paper is concerned with finite collections of sets in normal spaces (as contrasted with the special and more familiar case of compact metric spaces) and in particular with equivalences between finite coverings by open sets and finite coverings by closed sets. The main result is this: Let R be a normal space, $F_1, \cdots, F_s$ a finite multiplicative (additive) system of closed sets of R, and U_i, $i=1, \cdots, s$, any open sets for which $F_i \subset U_i$. Then one can find a system G_i, $i=1, \cdots, s$, of open sets satisfying $F_i \subset G_i \subset U_i$, equivalent to the F_i in the sense of preserving product (sum) and such that a sub-collection of the G_i covers R if and only if the corresponding sub-collection of the F_i does also. *H. Wallman* (Princeton, N. J.).

Kakutani, Shizuo. Weak topology, bicompact set and the principle of duality. Proc. Imp. Acad. Tokyo **16**, 63–67 (1940). [MF 2215]

The author announces a program for unifying various known duality and "realization" theorems, and proving new ones. In particular, conjugate Banach spaces, Pontrjagin's duality theorems for Abelian groups and the realization theorem of Stone for Boolean algebras by subsets of bicompact spaces are described as instances of a general "duality principle." The role of bicompactness in this principle is ascribed to a very general theorem of Tychonoff [Math. Ann. **111**, 762–766 (1935)]. As an application, the author defines abstract (M)-spaces conjugate to the abstract (L)-spaces introduced by the reviewer, and proves that any abstract (M)-space is isometric and lattice-isomorphic to the concrete (M)-space of all continuous real functions on a suitable bicompact topological space Ω, with $\|x\|=\max|x(t)|$. *G. Birkhoff* (Cambridge, Mass.).

Vedenissoff, N. Remarques sur la dimension des espaces topologiques. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 131–140 (1939). (Russian. French summary) [MF 2107]

Given an arbitrary T_1-space one says that it has dimension not greater than n (a) in the Menger-Urysohn sense if every point can be surrounded arbitrarily closely by neighborhoods with boundaries of dimension not greater than $n-1$, (b) in the Čech sense if every closed set can be surrounded arbitrarily closely by neighborhoods with boundaries of dimension not greater than $n-1$, (c) in the covering sense if every finite covering by open sets admits a refinement of order not greater than $n-1$, (-1)-dimensional meaning the empty set. It is shown on one hand that for $n=0$ senses (b) and (c) always coincide; on the other hand, making use of an example of Tychonoff, it is demonstrated that a T_1-space may have dimension 0 in sense (a) and a positive dimension in sense (b) or (c). As regards arbitrary n it is possible to prove that (a) and (b) coincide for perfectly normal spaces (normal spaces in which every open set is an F_σ) having the property that every covering by open sets admits a countable refinement; and for $n=0$, 1 the perfect normality may be weakened to regularity. *H. Wallman* (Princeton, N. J.).

Szpilrajn, E. Remarques sur l'ensemble de Lusin. Mathematica, Cluj **16**, 50–52 (1940). [MF 2482]

Let X be a metric space. A set E in X is a Lusin set if each subset of E which is non-dense in X is at most denumerable. The space X has the property (ν_0) if each closed set in X is the sum of an open set and a set which is at most denumerable. The space X has the property (ν_1) if each closed set in X is of the form $G-D_1+D_2$, where G is open and D_1, D_2 are at most denumerable. The author shows the

equivalence of the propositions: (i) There exists a non-denumerable, linear Lusin set. (ii) There exists a separable, non-denumerable space with the property (ν_0). (iii) There exists a separable, non-denumerable space with the property (ν_1). The proof that (iii) implies (i) is based on a theorem of Kuratowski-Sierpinski [Fund. Math. **26**, 137–142 (1936)].

L. W. Cohen (Lexington, Ky.).

Miller, Harlan C. A theorem concerning closed and compact point sets which lie in connected domains. Bull. Amer. Math. Soc. **46**, 848 (1940). [MF 2932]

It is shown that a closed compact subset of a connected domain is contained in a compact continuum lying in the domain. This result holds in any space which either satisfies Axioms 0, 1, 2 of R. L. Moore's Foundations, or satisfies Axioms 0, 1 and is locally arcwise connected.

H. M. Gehman (Buffalo, N. Y.).

Montgomery, Deane and Zippin, Leo. Topological transformation groups. I. Ann. of Math. (2) **41**, 778–791 (1940). [MF 3021]

G is a compact metric topological group. When G is a transformation group of a topological space E, it is said to be effective on $M\subset E$ if every element of g but the identity moves some point of M. The structure theorem for orbits, namely, if G acts effectively on the finite dimensional orbit $M=G(x)$ of a point x, then M is locally the topological product of a k-cell and a compact zero-dimensional set, is first proved for an n-dimensional connected G using the local decomposition of G into the direct product of a local Lie group L and a central subgroup Z of G. The restriction on the dimension of G is removed by showing that G has a connected invariant subgroup K idle on M and such that G/K is finite dimensional. This is done by using a sequence of invariant subgroups G_i shrinking towards the identity and such that G/G_i is a Lie group. Then since G finite dimensional on M for dim $M>0$ implies G has a one parameter subgroup operating homeomorphically on x, it follows that for some n the identity component K of G_n is idle on M. When G is finite dimensional and connected, the closed subgroup G^* which leaves x invariant is a Lie group, and M is locally connected if and only if G is a Lie group; dim G^*+dim $M=$dim G, which also holds when G is neither connected nor effective.

The orbit M is homeomorphic to the decomposition space [Alexandroff-Hopf, Topologie, p. 63] G/G^* of left cosets gG^*. If G is the limit of the inverse homeomorphism sequence (A-sequence) H_i, $H_i=f_i(H_{i+1})$, of Lie groups and if H_i^* is the closed subgroup of H_i associated with a closed subgroup G^* of G, continuous f_i^* are defined so that H_i/H_i^*, $H_i/H_i^*=f_i^*(H_{i+1}/H_{i+1}^*)$, has G/G^* as limit. In the course of the paper it is proved that dim $G/G^*=\lim d_i$, where $d_i=$dim H_i/H_i^*; d_i is non-decreasing; if G/G^* is locally connected and of finite dimension, f_i^* is a homeomorphism for large enough i. The proof of the last result by means of covering spaces permits the removal of the connectedness hypothesis in the orbit structure theorem. When G_n is a sequence of closed subgroups approaching the closed subgroup G, then, for large enough n, dim $G/G_n\geqq$dim G/G^* and if G acts on a space E the points on orbits of dim$\geqq k$ form an open set. For orbits N near a finite dimensional orbit M, K_N is a subgroup of K_M, where K_M is the component of the identity of that subgroup G_M which leaves all of M invariant (an example shows that G_N need not be a subgroup of G_M). Hence if E is locally Euclidean and connected, G a Lie group and G_M such that G/G_M is of highest dimension, then K_N and K_M are identical and K_M is idle on an open subset of E and [Newman, Quart. J. Math. **2**, 1–8 (1931)] K_M is idle on E. So if G is effective, K_M is the identity and G_M is finite. An induction on k now gives that G effective on M and dim $M=k$ imply dim $G\leqq\frac{1}{2}k(k+1)$. Use of an A-sequence removes the Lie restriction from this last result. Another argument involving K_M shows that, if G is effective on a connected space E and all orbits have dimension k, then G is finite dimensional. If G is given by an A-sequence and G_p is the subgroup of G consisting of those elements whose associated element in H_n is the identity for $n\leqq p$, then G_p is an invariant subgroup of G and $H_p=G/G_p$. If the orbit G/G^* is locally connected, $G_p\subset G^*$ for some p and G_p leaves the orbit pointwise invariant. So if G acts effectively on a locally connected orbit, $G=H_p$ is a Lie group. The set of points on locally connected orbits form an F_σ. If G acts effectively on a locally Euclidean space and all the orbits are locally connected (that is, each is a finite set of manifolds) then G is a Lie group. This is because $p(x)$ is bounded in E, where $p(x)$ is the least p such that G_p is in the subgroup leaving x invariant.

W. W. Flexner (Ithaca, N. Y.).

Tola, José P. Observation on operations continuous according to Cauchy and according to Heine in topological spaces. Actas Acad. Ci. Lima **3**, 43–47 (1940). (Spanish) [MF 2711]

Let L_x, L_y and L_z be Hausdorff spaces, the first two of which satisfy the first denumerability axiom. It is proved that a function $f(x, y)$ on L_x, L_y to L_z is continuous in the neighborhood sense if it is continuous in the sequential sense. An example is given to show that the denumerability axiom is not a necessary condition. The reviewer was unable to see how the example applied.

J. V. Wehausen.

Tola, José P. On the possibility of defining a continuous absolute value of the elements of a linear topological space. Actas Acad. Ci. Lima **3**, 29–30 (1940). (Spanish) [MF 2472]

Tola, José P. On the possibility of defining a continuous absolute value of the elements of a linear topological space. Revista Ci., Lima **43**, 303–324 (1940). (Spanish) [MF 2472]

A linear topological space will be said to have the property (P) if every neighborhood of the zero-element θ contains another neighborhood of θ such that no two elements of its frontier lie on the same ray originating at θ. It is proved that a necessary and sufficient condition for it to be possible to introduce into a linear topological space a homogeneous, continuous, non-negative absolute value $|x|$ which preserves the given topology is that the space be locally bounded and have the property (P). $|x|$ satisfies only the "weak" triangle condition, that is, for any $\eta>0$ there exists $\epsilon>0$ such that $|x|<\epsilon$, $|y|<\epsilon$ implies $|x+y|<\eta$. This result supplements one of Hyers [Revista Ci., Lima **41**, 555–574 (1939); these Rev. **1**, 318]. Hyers assumes only local boundedness and obtains only upper semi-continuity of $|x|$.

J. V. Wehausen (Columbia, Mo.).

Ayres, W. L. A note on the definition of arc-sets. Bull. Amer. Math. Soc. **46**, 794–796 (1940). [MF 2924]

The author proves that in order for a nondegenerate subset A of a Peano space P to be an A-set, that is, that it be closed and contain every arc of P whose end points it contains, either of the following conditions is necessary and

sufficient: (1) every set separating two points of A in A (in the weak sense) also separates them in P; (2) A be connected, $\bar{A}\cdot\bar{C}$ be a single point for each component C of $P-A$, and this point y belong to A if $\bar{A}$ contains two continua S and T such that $S\cdot T=y$. *G. T. Whyburn.*

Stoïlow, S. Des sous-ensembles sur lesquels une transformation continue d'un espace est transformation intérieure ou topologique. Disquisitiones Math. et Phys. Publ. Inst. Cercetări Ştiintifice Regele Carol II 1, 23–28 (1940). [MF 2869]

This paper considers a single-valued continuous transformation $f(X)=Y$, where X and Y are metric spaces. The transformation f is said to be non-singular on a subset K of X provided that for any open set K' contained in K the set $f(K')$ contains an open subset of Y. The author shows that if X is separable and locally compact and if Y is complete then there exist G_δ subsets A and B of X and Y, respectively, such that the following conditions are satisfied: (i) $f(A)=B$ is interior (in the sense that the image of every set open in A is a set open in B), (ii) B is residual in Y, (iii) A is residual in every open subset of Y on which f is non-singular. In the second part of the paper the author applies this theorem to topological varieties of dimension n, that is, to connected T_1 spaces [see Alexandroff and Hopf, Topologie I, Berlin, 1935, p. 50] having for each of their points a neighborhood homeomorphic with the n-dimensional Euclidean space. He considers the case where X and Y are separable topological varieties of dimension m and n, respectively, where f is non-singular on X, and where $f^{-1}(y)$ is denumerable for all y in a residual subset of Y. Under these conditions he shows that every open subset G of X contains a G_δ subset E of X such that $f(E)=F$ is topological and F is a G_δ subset of Y which is residual in some open subset of Y. *D. W. Hall* (Providence, R. I.).

Abe, Makoto. Über die stetigen Abbildungen der n-Sphäre in einen metrischen Raum. Jap. J. Math. 16, 169–176 (1940). [MF 2597]

It was shown by J. H. C. Whitehead [Proc. London Math. Soc. (2) 45, 279 (1939)] and the reviewer [Fund. Math. 32, 167–175 (1939)] that every element w of the fundamental group $\pi_1(Y)$ of the space Y determines an automorphism $w(\alpha)$ of the nth homotopy group $\pi_n(Y)$. If $n=1$ the inner automorphism $w(\alpha)=w\alpha w^{-1}$ is obtained. Modifying slightly the definition of $\pi_n(Y)$, Abe constructs a group $\kappa_n(Y)$ which contains $\pi_n(Y)$ and also a subgroup $\bar{\pi}_1(Y)$ isomorphic with $\pi_1(Y)$, and defines in a natural way a homomorphism h mapping $\kappa_n(Y)$ into $\bar{\pi}_1(Y)$ such that $h(\bar{w})=\bar{w}$ for every $\bar{w}\epsilon\bar{\pi}_1(Y)$ and that $h(\alpha)$ is the neutral element if and only if $\alpha\epsilon\pi_n(Y)$. Consequently $\pi_n(Y)$ is a self-conjugate subgroup of $\kappa_n(Y)$ and $\kappa_n(Y)/\pi_n(Y)\approx\pi_1(Y)$. Further Abe proves that $w(\alpha)=\bar{w}\alpha\bar{w}^{-1}$ for every $w\epsilon\pi_1(Y)$ and $\alpha\epsilon\pi_n(Y)$, thus showing that in the larger group $\kappa_n(Y)$ the automorphisms of $\pi_n(Y)$ induced by $\pi_1(Y)$ are all inner automorphisms. This permits a uniform treatment of the cases $n=1$ and $n>1$. It follows also that $\kappa_n(Y)$ is the direct product of $\pi_n(Y)$ and $\bar{\pi}_1(Y)$ if and only if $w(\alpha)=\alpha$ for every w and α, that is, when Y is n-simple in the sense of the reviewer [loc. cit.]. *S. Eilenberg* (Ann Arbor, Mich.).

Komatu, Atuo und Sakata, Ryoji. Einige Sätze über Abbildungen auf die Sphäre. Jap. J. Math. 16, 163–167 (1940). [MF 2596]

The authors generalize a result formerly proved by them for complexes only [Jap. J. Math. 16, 57–62 (1939); these Rev. 1, 106]. They show that a mapping of a compact space A of finite dimension in the sphere S^r is essential if and only if there exists a convergent cycle of A which is mapped essentially under f (it is meant of course that the set carrying the cycle is mapped essentially). Further the authors give a new proof for a result of Eilenberg-Freudenthal about extendibility properties of mappings of complexes on spheres of lower dimensions. *W. Hurewicz.*

Eilenberg, Samuel. On continuous mappings of manifolds into spheres. Ann. of Math. (2) 41, 662–673 (1940). [MF 2558]

The author establishes far reaching results in regard to continuous mappings of a finite and oriented $(n+1)$-dimensional manifold M^{n+1} into the sphere S^{m+1} of a lower dimension $(m<n)$. H. Hopf introduced two invariants for mappings of this kind. The first invariant $\Gamma(f)$ is an element of the $(n-m)$-dimensional homology group of M^{n+1} with integer coefficients (roughly speaking the inverse images of points of S^{m+1} can be regarded as $(n-m)$-cycles, and $\Gamma(f)$ is the homology class of these cycles). The second invariant $c(f)$ is defined only in the case $n=2m$, $\Gamma(f)=0$; $c(f)$ is the linkage coefficient of two m-cycles of M^{2m+1} corresponding to two distinct points of S^{m+1}. $\Gamma(f)$ and $c(f)$ are both homotopy invariants, that is, they do not change if f undergoes a continuous deformation; $\Gamma(f)$ and $c(f)$ are certainly 0 if there exists a point of S^{m+1} whose inverse image is a polytope of dimension less than $n-m$. The author proves the converse statement: If the condition $\Gamma(f)=0$ (and for $n=2m$ the additional condition $c(f)=0$) is satisfied, f is homotopic to a mapping g such that the inverse image of at least one point of S^{m+1} (even of at least two points) is an $(n-m-1)$-dimensional polytope. This result can be simplified if M^{n+1} is subject to the following homology condition: (H) The ith homology group of M^{n+1} with (m^{n-i}) as the coefficient group is 0. ((m^i) stands for the ith homotopy group of the m-sphere.)

If (H) is satisfied, then any mapping f with the property $\Gamma(f)=0$ (completed by $c(f)=0$ if $n=2m$) is homotopic to a mapping g such that the inverse image of at least one point of S^{m+1} under g is a single point. (H) is certainly satisfied in the following two cases: (1) $n=m+1$, (2) $M=S^{n+1}$. The last case leads to various consequences concerning mappings of spheres into spheres. The author investigates further the case $n=2m$, $\Gamma(f)=0$, $c(f)\neq0$. It turns out that, given two points p and q of S^{m+1} and two integers α_1 and α_2 with $\alpha_1\alpha_2=c(f)$, f is homotopic to a mapping g such that the cycles corresponding to the points p and q are essentially m-spheres with multiplicities α_1 and α_2. Using this result a problem of H. Hopf is solved by showing that for any two integers α_1, α_2 there exists a mapping of $S^m\times S^m$ in S^m of type (α_1, α_2), that is, a mapping transforming each sphere $S^m\times p$ with degree α_1 and each sphere $p\times S^m$ with degree $\alpha_2(p\epsilon S^m)$. *W. Hurewicz* (Chapel Hill, N. C.).

Cairns, Stewart S. Homeomorphisms between topological manifolds and analytic manifolds. Ann. of Math. (2) 41, 796–808 (1940). [MF 3023]

A topological m-manifold M is a connected topological space which can be covered by a denumerable set of neighborhoods each of which is an m-cell. If the transformation between the local coordinates on M where they overlap is analytic with non-vanishing Jacobian, M is analytic. If M can be triangulated it is known that it can be mapped into a simplicial complex P in an n-dimensional Euclidean space

E^n where the simplexes of P corresponding to the cells of M are Euclidean. A Brouwer manifold is a triangulated M having as the star of each of its vertices an m-cell which can be mapped homeomorphically into E^m so that the image of each component m-cell of the star is a simplex. M is in normal position if to each point of M there is a transversal varying continuously with the point. The principal theorems are: (1) To a topological manifold M there exists a set of coordinate systems in terms of which M is analytic with analytic Riemannian metric if and only if M can be triangulated so as to have a polyhedral representation in normal position. (2) Arbitrarily near any normal position of P there exists an analytic manifold homeomorphic to P. (3) If M is 3-dimensional and can be triangulated its P can be put in normal position. The proof of necessity in (1) follows from Whitney [Ann. of Math. (2) **37**, 645–680 (1936)] and Cairns [Ann. of Math. (2) **37**, 409–415 (1936)]. If P can be put in normal position it is a Brouwer manifold. A continuous family of transversal planes to a Brouwer P in normal position is constructed by induction such as to permit the proof of (2) (and so of the sufficiency of (1) through application of methods and results due to Whitney [loc. cit.]). If S^k represents the generic vertex star of a Brouwer k-manifold and $\Pi^{k\nu}$ is the space of all $(\nu-k)$-planes transversal to S^k in E^ν, a by-product of the induction is that every Brouwer m-manifold can be put in normal position provided that every singular $(m-k-1)$-sphere in $\Pi^{k\nu}$ bounds an $(m-k)$-cell, $0\leqq k<m$. Equivalent forms of this result are given in terms of "spaces of mappings" of S^k into E^n and "spaces of triangulations" of S^k. It is proved that every triangulated m-manifold, $m=3$, is a Brouwer manifold by showing that each triangulation of an $(m-1)$-sphere can be mapped homeomorphically into a geodesic triangulation for $m=3$. With this result, the inductive construction can be used when $m=3$ to show that every Brouwer 3-manifold can be put in normal position (thus proving (3)); and, when $m=4$, to show that the normal position problem is equivalent to finding, for every pair of geodesic triangulations of the 2-sphere which are homeomorphic with preservation of orientation, a continuous family of homeomorphic geodesic triangulations joining them. *W. W. Flexner.*

Dunford, Nelson. On continuous mapping. Ann. of Math. (2) **41**, 639–661 (1940). [MF 2557]

Let X be an arbitrary set, and let $\mathfrak{P}(X)$ be its power set. The main object of this paper is to establish sufficient conditions that a 1-1 continuous function f map a non-void open set $G\epsilon\mathfrak{P}(X)$ onto a set having at least one interior point. In the first part of the paper, a family $\mathfrak{N}$ of neighborhoods N is postulated with the properties: (A)(o) $\mathfrak{N}\subset\mathfrak{P}(X)$; (i) $N\epsilon\mathfrak{N}$ implies $N\neq 0$; (ii) $N_1N_2\neq 0$ implies the existence of an $N_3\subset N_1N_2$. Also, an operation $\bar{E}$ of closure on $\mathfrak{P}(X)$ to $\mathfrak{P}(X)$ is postulated with the properties: (B)(i) Any subset of a non-dense set is non-dense, where a non-dense set is one whose closure contains no neighborhood. (ii) $N\subset\bar{E}$ implies $N\subset\overline{NE}$. (iii) The void set is non-dense. In most of the paper it is further assumed that: (C′) The sum of two non-dense sets is non-dense. I_ϕ denotes the family of sets E in $\mathfrak{P}(X)$ expressible as a sum of ϕ non-dense sets, and $II_\phi=\mathfrak{P}(X)-I_\phi$. A set E is locally II_ϕ on a set e if $NE\epsilon II_\phi$ for every N with $Ne\neq 0$. $\mathfrak{B}_\phi$ denotes the family of sets E such that $N-E\epsilon I_\phi$ whenever E is locally II_ϕ on N. Author develops properties of the families I_ϕ, II_ϕ and $\mathfrak{B}_\phi$, generalizing well-known theorems relating to category, the Baire condition and the Souslin operation A; in particular, he shows the invariance of $\mathfrak{B}_\phi$ under a generalized operation A_{ϕ_ϵ}. In the second part of the paper, author establishes theorems of the following type, where f is a continuous function on a space X to a space X_1: If $f(\mathfrak{N})\subset\sum_\phi\mathfrak{B}_\phi II_\phi$, then $f(G)=G_1+E_1$, where G is any non-void open set in X, G_1 is a non-void open set in X_1 and E_1 is a non-dense set closed in $f(G)$, where various conditions are imposed on the spaces X and X_1, and where the function f, instead of being 1-1, is generalized into types called almost 1-1 (that is, $f(N)f(N')$ is non-dense whenever the neighborhoods N, N' are such that $NN'=0$), or distributive (that is, for any $p>0$ there exists $q>0$ such that, if x, x', N are such that $x\epsilon N\epsilon\mathfrak{N}$, diam $N<q$, $f(x)=f(x')$, then there exists N' such that $x'\epsilon N'\epsilon\mathfrak{N}$, diam $N'<p$, $f(N)=f(N')$). Author also establishes a condition that $f(\mathfrak{N})\subset\sum_\phi\mathfrak{B}_\phi II_\phi$ when X is a complete metric space. As an application, author establishes conditions that f take open sets into open sets when X and X_1 are certain topological groups. *C. C. Torrance.*

Polak, A. Stetige Abbildungen metrischer Räume und ihre Beziehungen zu den offenen Abbildungen. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 165–180 (1939). (Russian. German summary) [MF 2112]

Let X and Y be separable metric spaces and f a continuous mapping of X on Y. A point x of X is called regular with respect to f if the image of every neighborhood of x contains a neighborhood of $f(x)$. Suppose that Y is a complete space and f a closed mapping. The author shows under these assumptions (which are certainly satisfied if X is compact) that there is at least one regular point in X. Moreover the image of the set of regular points is dense in Y. The set M of regular points contains a set dense in M on which the mapping f is open. The author calls f a semi-open mapping if the image of every open set contains a non-vacuous open subset. Under the same assumptions as above the following statement holds: f is semi-open if and only if there exists a set dense in X on which f is open. Some further results concern mappings f having inverse images $f^{-1}(y)$ $(y\epsilon Y)$ nowhere dense in X.

W. Hurewicz (Chapel Hill, N. C.).

Blumenthal, Leonard M. and Thurman, George R. The characterization of pseudo-spherical sets. Amer. J. Math. **62**, 835–854 (1940). [MF 2884]

The authors consider the n-dimensional spherical surface of radius r (with distances measured along great circles), and determine some of those pseudo-metric sets P such that, while each $n+2$ points of P may be isometrically embedded in the spherical surface, not every $n+3$ points of P can be so embedded. Such a set is called a pseudo-$S_{n,r}$ set. It is known that, if every $n+3$ points of a set can be embedded in an n-dimensional spherical surface, then the whole set can be embedded in the surface. The discussion revolves about various determinants of the form $|\cos(p_ip_j/r)|$, and depends on the known connection of these determinants with the possibility of embedding the set $\{p_i\}$ in the spherical surface. It is proved that, if $\{p_i\}$ is a pseudo-$S_{n,r}$ set which (1) contains more than $n+3$ points, and (2) contains no pair of points at distance πr, then $\cos(p_ip_j/r)=-\epsilon_i\epsilon_j/(n+1)$ for $i\neq j$, where $\epsilon_i=\pm 1$. Thus these pseudo-$S_{n,r}$ sets are closely related to certain equilateral sets. The results may be extended to other spaces in which a monotone function plays the role of $\cos(x/r)$, in particular to the spherical surfaces with distances measured along chords. *J. W. Tukey.*

Jones, F. B. Almost cyclic elements and simple links of a continuous curve. Bull. Amer. Math. Soc. 46, 775–783 (1940). [MF 2922]

Much of the work that has been done in the theory of cyclic elements of continuous curves requires that the set be locally compact. Without this requirement, it does not necessarily hold that every simple closed curve belongs to a cyclic element of the curve, where a cyclic element is defined as either a cut point or a set M_p, where p is a non-cut point and M_p is the set of all points which are not separated from p by any point. The author of the present paper works in a space satisfying R. L. Moore's axioms 0 and 1. He defines a "cyclic nucleus" K of a continuous curve. He shows, among other properties, that neither K nor $\bar{K}$ contains a cut point of itself and that, if J is a simple closed curve of M, then there is one, and only one, cyclic nucleus containing J. He further defines "almost cyclic elements" H and shows that, if an almost cyclic element of a continuous curve M contains a proper point of M, then H is a simple link of M. He gives an example of an almost cyclic element containing no proper point and makes interesting observations concerning the relation of Axiom 3 to Axioms 0, 1, 2, and 4 of R. L. Moore. *J. R. Kline* (Philadelphia, Pa.).

Harrold, O. G., Jr. A note on strongly irreducible maps of an interval. Duke Math. J. 6, 750–752 (1940). [MF 2728]

The author proves that, if the set N of all non-local separating points of a locally connected continuum M is dense in M and if P is any countable subset of N which is dense in N, there exists a continuous mapping $f(I)=M$ of the unit interval I onto M such that $f^{-1}(y)$ is single valued for each $y\epsilon P$ and such that $f^{-1}(P)$ is dense in I. Thus, under these conditions, there exists a strongly irreducible mapping of I onto M, that is, a mapping such that no closed proper subset of I maps onto all of M. *G. T. Whyburn.*

Steenrod, N. E. Regular cycles of compact metric spaces. Ann. of Math. (2) 41, 833–851 (1940). [MF 3026]

A regular cycle of a compact metric space X is essentially a locally finite, at most countable, cycle with vertices in X and with the diameters of the simplices tending to zero. These cycles lead to a homology group $H^q(X)$ which is a new invariant. A subgroup $\tilde{H}^q(X)$ of $H^q(X)$ is defined by considering regular cycles which are the sums of sequences of finite cycles. If the coefficient domain is compact, then $\tilde{H}^q(X)=0$. In general $H^q(X)-\tilde{H}^q(X)\approx V^{q-1}(X)$, where V^{q-1} is the $(q-1)$-Vietoris group of X with the same coefficient domain. A method is given which permits to deduce $\tilde{H}^q(X)$, with any coefficient domain, from the group $V^q(X)$ with coefficients mod 1, and it is shown that $\tilde{H}^q(X)=0$ if X is sufficiently "smooth." The method is applied to the case when $X=\Sigma$ is a solenoid: it is shown that $H^1(\Sigma)=\tilde{H}^1(\Sigma)$ (integer coefficients) has uncountably many elements of infinite order.

If X is imbedded into S^n then S^n-X is subdivided into an infinite complex K and the groups Z^q, B^q and $\bar{B}^q$ of infinite cycles, bounding infinite cycles and limits of bounding infinite cycles can be defined. The homology group $H^q(K)=Z^q-B^q$ is considered, and its subgroup $\tilde{H}^q(K)$ is defined in a way analogous to $\tilde{H}^q(X)$. The main duality theorem is: $H^q(K)\approx H^q(X)$, $\tilde{H}^q(K)\approx\tilde{H}^q(X)$. This shows for the first time that $H^q(K)$ and $\tilde{H}^q(K)$ are topological invariants of X alone. If the coefficient domain is a division-closure group, then it is shown that $\tilde{H}^q(K)=\bar{B}^q-B^q$ and consequently $Z^q-\bar{B}^q\approx V^{q-1}(X)$. This duality theorem, due to Lefschetz, and the complete lack of information about the group Z^q-B^q were the reasons why $Z^q-\bar{B}^q$ was accepted as a standard definition of the homology group of K. The results are applied to prove some theorems about the mappings of S^n-X into S^m. In particular, if $X=\Sigma$ is a solenoid, it is shown that the set of all homotopy classes of maps of $S^n-\Sigma$ into S^{n-1} is uncountable. *S. Eilenberg* (Ann Arbor, Mich.).

Whitehead, J. H. C. On C^1-complexes. Ann. of Math. (2) 41, 809–824 (1940). [MF 3024]

The object is stated to be the provision of a foundation for theorems involving both differential geometry and combinatorial equivalence [cf. the paper reviewed below]. The work of S. S. Cairns [Ann. of Math. (2) 35, 579–587 (1934) and 37, 409–415 (1936)] is relevant. Denote points of Euclidean metric spaces R^k and R^n by $x=(x^1,\cdots,x^k)$ and $y=(y^1,\cdots,y^n)$, respectively. Let K be a finite, rectilinear, simplicial complex in R^k. A continuous map $y=f(x)$ of K into R^n is called a C^1-complex if to each simplex A of K there is a neighborhood U in R^k and a continuous map g of U into R^n such that $g(x)=f(x)$ for every $x\epsilon A$ and $\partial g^i/\partial x^j$ $(i=1,\cdots,n;\ j=1,\cdots,k)$ exist and are continuous in U.

Let A be a simplex of K and b a point of A. The map φ of A into R^n which is linear and satisfies the two conditions $\varphi(b)=f(b)$ and $d\varphi(b)=df(b)$ is called the tangent simplex at $f(b)$ to $f(A)$. The set of tangent simplexes at $f(b)$ is called the tangent star at $f(b)$ to the C^1-complex $f(K)$. A C^1-complex f is non-singular if it is $1:1$ and if the obvious mapping of the star of b in K onto the tangent star at $f(b)$ is also $1:1$. The Jacobian determinant $\partial f/\partial x$ of a non-singular C^1-complex f does not vanish. The map of K into R^n which is linear and coincides with f at the vertices of K is denoted by $L_f(K)$. The author proves that, given a non-singular C^1-complex f, there are subdivisions $K^{(i)}$ of the antecedent K such that $L^i=L_f(K^{(i)})$ are non-singular and converge to f in the sense that $L^i(x)\to f(x)$ and

$$dL^i(x)/\|df(x)\|\to df(x)/\|df(x)\|$$

uniformly in x and dx. A non-singular C^1-complex f whose image is an n-dimensional manifold M^n of class C^1 is called a C^1-triangulation of M^n. If $f(K)$ is a C^1-triangulation of M^n then K is a formal manifold, that is, the complement of every vertex in K is combinatorially equivalent to the boundary of an n-simplex. Every manifold of class C^1 has a C^1-triangulation. If $f_1(K_1)$ and $f_2(K_2)$ are C^1-triangulations of a manifold of class C^1, then K_1 and K_2 are combinatorially equivalent. The author proves several other theorems of a similar nature. Some of the results are extended to infinite complexes and open manifolds. The notion of a C^λ-complex $(\lambda>1)$ is considered briefly. *R. H. Fox* (Urbana, Ill.).

Whitehead, J. H. C. On the homotopy type of manifolds. Ann. of Math. (2) 41, 825–832 (1940). [MF 3025]

An n-dimensional formal manifold [see the preceding review] M^n is said to belong to class II if there is an integer k such that the product of M^n and a k-simplex A^k can be rectilinearly imbedded in the Euclidean $(n+k)$-space R^{n+k}. The author's definition of class II is equivalent but more involved; furthermore, the above definition need not be restricted to manifolds but can be applied to any complex. The property of belonging to class II is a combinatorial invariant of M^n. A 2-dimensional closed manifold belongs to II if and only if it is orientable. The operation of removing

from a simplicial complex a principal open simplex which has a free edge is called an elementary contraction. A contraction is a sequence of elementary contractions. If a complex K contains manifolds M_1^n and M_2^n of class II and if K contracts into M_1^n and also into M_2^n then, for any sufficiently large k, $M_1^n \times A^k \equiv M_2^n \times A^k$ [$\equiv$ denotes combinatorial equivalence]. The conditions of this theorem are satisfied when M_1^n and M_2^n belong to the same homotopy type and their fundamental group is of a certain character [for example, cyclic group of order 2 or 3]. In this theorem M_1^n and M_2^n need not be manifolds if the reviewer's definition of class II is adopted [see above]. If M^n is a smooth manifold (that is, of class C^1) which is connected, simply connected and acyclic in every dimension, then $M^n \epsilon \mathrm{II}$ and $M^n \times A^{n+4} \equiv A^{2n+4}$. The Poincaré hypothesis, for smooth manifolds at least, is contained in the hypothesis: If $M^p \times A^1 \equiv A^{p+1}$, then $M^p \equiv A^p$. Examples are given of topologically distinct manifolds M_1^n and M_2^n such that $M_1^n \times A^1 \equiv M_2^n \times A^1$. The reviewer notes that it can be shown that closed, connected 2-dimensional manifolds M_1^2 and M_2^2 have the property $M_1^2 \times A^1 \equiv M_2^2 \times A^1$ if and only if M_1^2 and M_2^2 have the same orientability and Euler characteristic. A smooth manifold belongs to class II if its normal sphere space in some Euclidean space is simple [in the sense of H. Whitney]. From this is deduced that a smooth orientable manifold belongs to the class II in each of the following three cases: (1) M^n is closed and admits an internal parallelism, as is always the case if $n=3$ or if M^n is a Lie group; (2) M^n is closed and can be represented as a manifold of class C^2 in R^{n+1} or in R^{n+2}; (3) M^n is bounded and all its cohomology groups vanish with integral coefficients.

In an appendix extracted from a letter to Hassler Whitney, the author gives an ingenious proof that the rotation group [of the unit sphere in Euclidean $(m+1)$-space] is simple in every dimension [in the sense of S. Eilenberg, Fund. Math. **32**, 167–175 (1939)]. However, it can be proved directly from the definitions that this is true of any connected space admitting a multiplication which is a continuous function and has a two-sided identity. Several applications of this result to Whitney's theory of sphere-spaces are given. *R. H. Fox* (Urbana, Ill.).

Cairns, Stewart S. Triangulated manifolds which are not Brouwer manifolds. Ann. of Math. (2) **41**, 792–795 (1940). [MF 3022]

Let a "star m-manifold" be a manifold on which the region covered by the star of any vertex is an m-cell. Let a "Brouwer m-manifold" be one in which each star has a rectilinear image in m-space E^m. Using an "essentially curvilinear triangulation" of an m-cell ($m \geqq 3$), the writer shows that there are manifolds of the first type which are not of the second type, for $m \geqq 4$. The second definition is not invariant under subdivisions. m-spheres ($m \geqq 3$) can be triangulated so that they have no convex image in E^{m+1}. There are complexes K such that the smallest n for which K has a rectilinear image in E^n is not invariant under subdivisions of K. *H. Whitney* (Cambridge, Mass.).

Freudenthal, Hans. Die Triangulation der differenzierbaren Mannigfaltigkeiten. Nachtrag. Nederl. Akad. Wetensch., Proc. **43**, 619 (1940). [MF 3104]

In connection with his paper in the same Proc. **42**, 880–901 (1939) [cf. these Rev. **1**, 106], the writer notes that the triangulation theorem was first proved by S. S. Cairns [Ann. of Math. (2) **35**, 579–587 (1934) and Bull. Amer. Math. Soc. **41**, 549–552 (1935)]. *H. Whitney.*

Komatu, Atuo. Über die Überdeckungen von Zellenräumen. III. Proc. Imp. Acad. Tokyo **16**, 55–58 (1940). [MF 2213]

This note outlines a proof that the homology groups of a Reidemeister "covering" U of a cell complex K depend only on the representation determined by U of the fundamental group of K in the group of automorphisms of the coefficient group of U [see Reidemeister, J. Reine Angew. Math. **137**, 164–173 (1935), or Topologie der Polyeder, 1938]. *A. W. Tucker* (Princeton, N. J.).

Flexner, W. W. Simplicial intersection chains for an abstract complex. Bull. Amer. Math. Soc. **46**, 523–524 (1940). [MF 2423]

This note defines the intersection $x \cdot z^*$ of a p-cell x of an abstract complex K and an $(r-p)$-cell z^* of the dual complex K^* as the multilinear form got by summing

$$[x:y_1][y_1:y_2]\cdots[y_{r-1}:z](x)(y_1)(y_2)\cdots(y_{r-1})(z)$$

for all y's such that $x>y_1>y_2>\cdots>y_{r-1}>z$. This is an r-chain of the first derived; (x), etc. are the "centroid" vertices corresponding to the cells x, etc. and $[x:y_1]$, etc. are incidence numbers of K. Then, as desired,

$$F(x\cdot z^*)=(Fx)\cdot z^*+(-1)^p x\cdot(Fz^*),$$

assuming that the cells of K^* are oriented so that $[y^*:x^*] = (-1)^{p+1}[x:y]$. *A. W. Tucker* (Princeton, N. J.).

de Rham, Georges. Sur les complexes avec automorphismes. Comment. Math. Helv. **12**, 191–211 (1940). [MF 1569]

L'auteur généralise et approfondit les recherches de Reidemeister [Abh. Math. Sem. Hansischen Univ. **11**, 102–109 (1935)], de Franz [J. Reine Angew. Math. **173**, 245–254 (1935)] et de lui-même [Rec. Math. [Mat. Sbornik] N.S. **1 (43)**, 737–742 (1936)] sur les nouveaux invariants topologiques joignant les notions d'homotopie aux notions d'homologie. Il considère des complexes à automorphismes, et il forme des chaînes de cellules dont les coefficients sont des éléments de l'algèbre A de G (le groupe des automorphismes) sur D (un anneau donné); si γ est un automorphisme de G et c est une chaîne, γc désigne la chaîne se déduisant de c par γ. Les groupes de Betti, qui sont des A-modules, se dérivent d'une manière évidente de cette notion. Au moyen d'un idéal premier $\mathfrak{p}$ de A on peut définir des groupes de Betti mod $\mathfrak{p}$, qui sont des $A_\mathfrak{p}$-modules ($A_\mathfrak{p}$ est l'anneau A mod $\mathfrak{p}$). Si, spécialement, $\mathfrak{p}$ est diviseur de l'idéal-commutateur de A, $A_\mathfrak{p}$ se réduit à un corps commutatif et les groupes de Betti mod $\mathfrak{p}$ deviennent des espaces vectoriels. On trouve un analogue de la formule d'Euler-Poincaré (*) $C^n+B^{n-1}+C^{n-2}+\cdots$ isomorphe à $B^n+C^{n-1}+B^{n-2}+\cdots$ où les C^q et B^q sont les groupes des chaînes et les groupes de Betti de q dimensions.

Ayant introduit une notion convenable de base pour les complexes à automorphismes et ayant précisé la notion d'équivalence pour les schémas d'incidence, on démontre que deux complexes homéomorphes au sens combinatoire ont des schémas équivalents et par suite des groupes de Betti A-isomorphes, respectivement $A_\mathfrak{p}$-isomorphes. Pour gagner de nouveaux invariants dans le cas où les groupes de Betti ne suffisent pas, l'auteur compare les déterminants de l'isomorphisme (*) (calculés au moyen de bases bien déterminées). Si les schémas des complexes sont équivalents l'un à l'autre, le quotient de ces déterminants (ressemblant à la torsion définie par Franz) doit être de la forme $\pm\gamma$ mod $\mathfrak{p}$, où γ est un élément de G indépendent de $\mathfrak{p}$. *H. Freudenthal* (Amsterdam).

Eilenberg, Samuel. On homotopy groups. Proc. Nat. Acad. Sci. U. S. A. **26**, 563–565 (1940). [MF 2707]

Let Y be a topological space. The author considers a continuous mapping f of a finite polytope K in Y such that the subpolytope K_m of K consisting of cells of dimensions not greater than m is carried into a single point y_0. If A^n is an n-chain in K (with coefficients from an abelian group G), the author calls the triple (K, A^n, f) an (n, m)-chain, and defines a new homology group $\mathfrak{H}^{n,m}(Y, G)$ using the (n, m)-chain and the $(n+1, m)$-chains. He establishes the following relation between the groups $\mathfrak{H}^{n,m}$ and the homotopy groups $\pi_i(Y)$: If $\pi_i(Y)=0$ for $i<m$, then $\mathfrak{H}^{n,m}(Y, G)$ is isomorphic to the nth homology group of Y defined in terms of ordinary continuous chains (that is, triples (K, A^n, f) without any restrictions upon f). Furthermore he defines a homomorphism of the group $\mathfrak{H}^{n,n}(Y, T)$ in the group $\pi_n(Y)$ ($n\geqq 1$, T=additive group of integers). For $n=1$ this gives the well-known homomorphism of the Poincaré group in the first homology group. *W. Hurewicz.*

Mayer, W. and Campbell, A. D. Generalized homology groups. Proc. Nat. Acad. Sci. U. S. A. **26**, 655–656 (1940). [MF 3172]

This preliminary report concerns a new homology theory constructed for every $p=2, 3, \cdots$. For $p=2$ the ordinary mod 2 theory is obtained. For $p=3$ the boundary operator $B(\ )$ satisfies $B\{B[B(\ )]\}=0$ instead of the usual $B[B(\ )]=0$. *S. Eilenberg* (Ann Arbor, Mich.).

Harrold, O. G., Jr. Exactly $(k, 1)$ transformations on connected linear graphs. Amer. J. Math. **62**, 823–834 (1940). [MF 2883]

A study is made of continuous transformations $f(A)=B$ such that, for each $y\epsilon B$, $f^{-1}(y)$ consists of exactly k points, where A is a connected graph (or in some cases a more general set) and k is an integer. Such a mapping is said to be exactly $(k, 1)$. It is shown, for example, that, if A and B are continua and either of them is a stably regular curve, so also is the other. Also, if $k>1$ and A is a continuum, B cannot be a simple arc. An example is given showing that the image of a graph need not be a graph under an exactly $(3, 1)$ mapping. If A is a connected graph and $f(A)=B$ is exactly $(k, 1)$, there is a closed set D of dimension 0 in B such that each component Q of $B-D$ is a free arc whose inverse consists of k open free arcs in A each mapping topologically onto Q; further, if $k=2$, B is a graph and there exist subdivisions of A and B into finite complexes K_A and K_B such that the mapping of K_A onto K_B is simplicial, and for any point y in B, the Menger-Urysohn order of y equals the arithmetic mean of the orders of its inverse points. Finally, a complete analysis of the possible topological types of $(2, 1)$ mappings on a simple closed curve is given. *G. T. Whyburn* (Charlottesville, Va.).

Veress, Pál. Über nicht-ebene Graphen. Mat. Fiz. Lapok **47**, 34–47 (1940). (Hungarian. German summary) [MF 2652]

Es wird in graphentheoretischer Fassung folgender Satz von Kuratowski bewiesen: es gibt nur zwei irreduzibile nicht-ebene Graphen, und zwar I. der vollständige Graph mit fünf Knotenpunkten, II. der Graph, der sechs Knotenpunkte A_1, A_2, A_3, B_1, B_2, B_3 besitzt und in dem jeder A-Punkt mit jedem B-Punkt durch eine Kante verbunden ist. *Author's summary.*

Scorza Dragoni, G. Un'osservazione sull'esistenza di elementi uniti nelle trasformazioni topologiche del cerchio. Ann. Mat. Pura Appl. (4) **19**, 45–49 (1940). [MF 3044]

This note proves the following theorem: A topological transformation t of a circle C into a set Γ of the plane possesses a fixed point if $C\cdot\Gamma$ contains at least two points and if every point of c, the boundary of C, which can not be joined to infinity without cutting either c again or the boundary of Γ is carried into a point of C. *D. Montgomery* (Northampton, Mass.).

Hirsch, Guy. Détermination du nombre algébrique des points fixes de certaines représentations. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **25**, 319–328 (1939). [MF 2541]

The polyhedra D_r^n considered are obtained by taking an n-dimensional pseudo-manifold having the Betti numbers of the n-sphere and removing the interior of $r+1$ disjoint n-cells. Continuous mappings T of D_r^n into itself are considered which, up to a homotopy, permute the $n+1$ boundary spheres of D_r^n. The degree $c(T)$ is defined and the algebraic number of fixed-points is shown to be $(-1)^n-(f-1)c(T)$, where f is the number of boundary spheres of D_r^n which T preserves globally, up to a homotopy. This number is 0 only if either $f=0$ and $c(T)=(-1)^{n+1}$ or $f=2$ and $c(T)=(-1)^n$. *S. Eilenberg.*

Mayer, W. A new approach to the critical value theory. Bull. Amer. Math. Soc. **46**, 838–847 (1940). [MF 2931]

Let Σ_m be a topological space, and Σ_k, $k=0, 1, \cdots, m-1$, subsets of Σ_m satisfying $\Sigma_m\supset\Sigma_{m-1}\supset\cdots\supset\Sigma_0$. Considering singular chains modulo 2, let $B_i(\Sigma_k)$ and $B_i(\Sigma_k-\Sigma_{k-1})$ be the i-dimensional Betti groups for Σ_k and Σ_k mod Σ_{k-1}, respectively. By expressing $B_i(\Sigma_k-\Sigma_{k-1})$ in terms of $B_i(\Sigma_k)$, $B_i(\Sigma_{k-1})$, and additional terms in a well-known manner, the author obtains the Morse inequalities $M_0\geqq R_0$, $M_1-M_0\geqq R_1-R_0$, etc., where

$$M_i=\sum_{k=0}^{m} rB_i(\Sigma_k-\Sigma_{k-1}),\quad R_i=rB_i(\Sigma_m),$$

and r denotes the rank of the group which follows. To relate this to the Morse theory, let Σ_m be a closed Riemann manifold, and J a twice differentiable function with only a finite number of stationary points. Define Σ_k as the set of all points for which $J\leqq\sigma_k$, where σ_k is the $(k+1)$st stationary value of J. The Morse type numbers for the stationary level σ_k are the quantities $rB_i(\Sigma_k-\Sigma_k')$ where Σ_k' is the point set $\{J<\sigma_k\}$. But, by J-deformations used in Seifert-Threlfall, "Variationsrechnung im Grossen," the groups $B_i(\Sigma_k-\Sigma_k')$ and $B_i(\Sigma_k-\Sigma_{k-1})$ are isomorphic. Consequently the quantities M_i defined above are identical with the Morse numbers. This method is then extended to the case when the space Σ is subdivided into a countable number of subsets Σ_k. *M. Shiffman* (New York, N. Y.).

ANALYSIS

Caton, Willis B. A class of inequalities. Duke Math. J. **6**, 442–461 (1940). [MF 2329]

Let $0<p_i<\infty$, $\sum p_i^{-1}=1$, $i=1, 2, 3$; α_1, α_2, β_1, $\beta_2\geqq 0$, $\alpha_1+\alpha_2=1$, $\beta_1+\beta_2=\beta_3$. The main purpose of the paper is the determination of the least upper bound K of

$$\sum_{n=1}^{\infty} a_n\Big\{\sum_{n=1}^{\infty}(n^{\beta_1}a_n^{\alpha_1})^{p_1}\Big\}^{-1/p_1}\Big\{\sum_{n=1}^{\infty}(n^{\beta_2}a_n^{\alpha_2})^{p_2}\Big\}^{-1/p_2}$$

for all $a_n \geqq 0$, $a_n \neq 0$. This is done under the condition $\alpha_1 p_1 = \alpha_2 p_2$, $\beta_1 p_1 > \beta_2 p_2$. In the second part of the proof for this result, a method of Gabriel is used. Further let $\alpha_1 p_1 - 1 > \alpha_2 p_2 - 1 > 0$. In case $\beta_1 p_1 \neq \alpha_1 p_1 - 1$, the existence of a finite K is proved, whereas for $\beta_1 p_1 = \alpha_1 p_1 - 1$, no finite K exists. *G. Szegö* (Stanford University, Calif.).

Popoviciu, Tiberiu. Notes sur les fonctions convexes d'ordre supérieur. III. Mathematica, Cluj **16**, 74–86 (1940). [MF 2487]

[Notes VI, VII and VIII have appeared earlier; cf. these Rev. **1**, 204; cf. also these Rev. **1**, 71 and the following review.]

By a slight extension of a theorem of Schur, Hardy, Littlewood and Pólya [Inequalities, Cambridge, 1934, p. 89, Theorem 108] the author proves the following: The inequality

$$(1) \qquad \sum_{j=1}^{r} \alpha_j f(x_j) \geqq \sum_{i=1}^{s} \beta_i f(y_i), \qquad r+s \geqq 3,$$

where $\alpha_j > 0$, $\beta_i > 0$, $\alpha_1 + \cdots + \alpha_r = \beta_1 + \cdots + \beta_s$, $x_1 < x_2 < \cdots < x_r$, $y_1 < \cdots < y_s$, is verified for any convex function $f(x)$ if and only if there are rs numbers $p_{ij} \geqq 0$ such that $\sum_{j=1}^{r} p_{ij} = 1$ $(i=1, \cdots, s)$, $\alpha_j = \sum_{i=1}^{s} \beta_i p_{ij}$ $(j=1, \cdots, r)$, $y_i = p_{i1} x_1 + \cdots + p_{ir} x_r$ $(i=1, \cdots, s)$. This furnishes elegantly as a special case the following theorem of K. Toda [J. Sci. Hiroshima Univ. Ser. A. **4**, 27–40 (1934)]: If $y_1, \cdots, y_{m-1}$ are the roots of the derivative of a polynomial of real roots $x_1, \cdots, x_m$, then

$$(2) \qquad \sum_{i=1}^{m} f(x_i)/m \geqq \sum_{i=1}^{m-1} f(y_i)/(m-1),$$

provided $f(x)$ is convex. The discussion of (1) is further extended to the case of convex functions of higher order with an application to the following problem of Toda: If $P(x) = (x-x_1) \cdots (x-x_m)$, x_r real, let $M_f(P) = \sum_1^m f(x_r)/m$. Is it true that the expression

$$F_n f(P) = \binom{n}{0} M_f(P) - \binom{n}{1} M_f(P') + \cdots + (-1)^n \binom{n}{n} M_f(P^{(n)})$$

is always positive or zero provided $f(x)$ is non-concave of order n? [For $n=1$ this is true and reduces to (2).] The author shows that there is no such theorem if $n>1$.

I. J. Schoenberg (Waterville, Me.).

Popoviciu, Tiberiu. Notes sur les fonctions convexes d'ordre supérieur (V). Acad. Roum. Bull. Sect. Sci. **22**, 351–356 (1940). [MF 2599]

The discussion of the extension of (1) [see preceding review] to convex functions of higher order is carried out here in greater detail and further extended to the case of inequalities of the type

$$(3) \qquad \sum_{i=1}^{m} \sum_{j=0}^{k_i} p_{ij} f^{(j)}(x_i) \geqq 0, \qquad 0 \leqq k_i \leqq n,$$

involving also derivatives of $f(x)$. By a direct argument the following inequality of this type is established: If x_i, y_i have the meaning they had in (2), then

$$(4) \qquad (m-1)[x_1, x_2, \cdots, x_m; f] \geqq [y_1, y_2, \cdots, y_{m-1}; f']$$

(the bracketed expressions are divided differences) provided $f(x)$ is non-concave of order n in an interval containing the x_i in its interior. A repeated application of (4) to the successive derivatives of $P(x) = \Pi(x-x_i)$ and subsequent choice of $f(x) = x^{n+r-1}$ $(r=1, 2, \cdots)$ leads to an interesting set of inequalities among certain symmetric functions of the roots of $P(x)$, $P'(x)$, $P''(x)$, $\cdots$, for the statement of which we refer to the paper. *I. J. Schoenberg* (Waterville, Me.).

Friedman, Bernard. A note on convex functions. Bull. Amer. Math. Soc. **46**, 473–474 (1940). [MF 2414]

A convex function of one variable is absolutely continuous. That this is not true for functions of two variables is shown by the author by the following example. Let $m(x)$ be Cantor's middle third function $(0 \leqq x \leqq 1)$, $f(x) = \int_0^x m(t)dt$. Then the function of two variables $f(x+y)$ is convex but not absolutely continuous, as is readily seen from the characteristic integral representation of such functions [Carathéodory, Reelle Funktionen, 1918, p. 654].

I. J. Schoenberg (Waterville, Me.).

Montel, Paul. Harmonic and subharmonic functions. Publ. Inst. Mat. Univ. Nac. Litoral **2**, 1–23 (1940). (Spanish) [MF 2511]

This is an expository paper dealing with some of the simpler aspects of the recent extensive work on convex functions of a single variable and on harmonic and subharmonic functions of two or more variables.

T. Radó (Columbus, Ohio).

Nicolesco, Miron. Continuité et dérivation polydimensionnelle et Laplacienne des suites. Rev. Math. Union Interbalkan. **3**, 1–16 (1940). [MF 2672]

It is shown that uniform convergence preserves the properties of mean circular continuity, and multidimensional continuity in the sense of K. Bögel, of functions of several variables. For the corresponding generalizations of differentiation, namely the generalized Laplacian operator and Bögel's multidimensional derivative, theorems on termwise differentiation of sequences are proved which are extensions of a result of O. Frink. In the case of the Bögel derivative, it is shown that the derived sequence of a convergent sequence, if it is uniform, must converge, and to the proper limit. The corresponding theorem for the generalized Laplacian requires in addition to uniformity a condition allowing the interchange of limit and integral. The following extension of a theorem of Frink is given: if a sequence is uniform at every point of a domain R, it converges uniformly in R if it converges at a single point of R. However, this is false, since any sequence of equicontinuous functions is uniform, and such a sequence may converge only at a single point. *O. Frink* (State College, Pa.).

Giraud, Georges. Sur quelques questions relatives aux intégrales convergentes et aux intégrales divergentes. J. Math. Pures Appl. **19**, 133–142 (1940). [MF 2362]

Let $\phi(x)$ be a given positive, increasing and unbounded function of $x>0$. Inspired by Borel's lectures on series with positive terms [Paris, 1902], the author constructs positive and continuous functions $f(x)$ and $g(x)$ such that $g(x) \leqq \phi(x) f(x)$, $\int_1^\infty f(t)dt$ converges, $\int_1^\infty g(t)dt$ diverges, and subject to various additional requirements concerning the behavior of $f(x)$ and $g(x)$. Six different sets of additional requirements are considered in turn. The first one requires both $f(x)$ and $f(x)/g(x)$ to be decreasing. The last requirement is as follows. Given are constants h and k such that $0<h<1<k$, and an integer $p>0$. The functions f, g to be constructed should be p times differentiable, $g^{(n)}(x)/f^{(n)}(x)$ and $(-x)^n f^{(n)}(x) \cdot x^k$ $(n=0, 1, \cdots, p)$ should be positive and increasing, while $(-x)^n g^{(n)}(x) \cdot x^h$ $(n=0, \cdots, p)$ should be decreasing. Extensive and ingenious use is made of the

weighted integral mean

$$\Phi(q, \lambda, x; f) = \frac{\lambda^q}{\Gamma(q)x^\lambda}\int_0^x f(t)(\log(xt^{-1}))^{q-1}t^{\lambda-1}dt$$

also recently used by F. W. Perkins [Amer. J. Math. 61, 217–230 (1939)] and the author in the theory of elliptic partial differential equations. *I. J. Schoenberg.*

Robertson, Fred. The general differential operator. Iowa State Coll. J. Sci. 14, 261–266 (1940). [MF 2656]

Formal relations are given employing a finite Laplace transformation and concerning fractional order integration and differentiation. *J. L. Barnes* (Medford, Mass.).

Leighton, Walter. Proper continued fractions. Amer. Math. Monthly 47, 274–280 (1940). [MF 2240]

The terminating or non-terminating continued fraction

$$(1)\qquad y_0 \sim b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \cdots$$

is defined to be the "proper" continued fraction generated by the real number y_0 with respect to the arbitrary sequence

$$(2)\qquad a_1, a_2, a_3, \cdots$$

of positive integers in case $b_k = [y_k]$, $y_k = a_k/(y_{k-1} - b_{k-1})$ $(k = 0, 1, 2, \cdots)$. It is proved that every proper continued fraction converges to the number y_0 generating it and that, with respect to a given sequence (2), a continued fraction of the form (1) is proper if and only if $a_n \leqq b_n$, where the upper sign holds between the final a and b in case (1) terminates. These results, well known for the regular continued fraction $(a_i = 1)$, generalize those for the continued cotangent of Lehmer in which the a's are not preassigned but depend upon the b's by the relation

$$a_n = 1 + (b_0 + b_1 + \cdots + b_{n-1})^2.$$

Every proper continued fraction converges more rapidly than a geometric series of ratio 1/2. *D. H. Lehmer.*

Hummel, P. M. Continued fractions and matrices. Tôhoku Math. J. 46, 340–359 (1940). [MF 2447]

Let $u_1' \neq 0, u_1'', \cdots, u_1^{(n)}$ be any n real numbers. The n-ary continued fraction representation of the $n-1$ ratios $u_1''/u_1', u_1'''/u_1', \cdots, u_1^{(n)}/u_1'$ is defined to be the sequence of partial quotient sets of integers

$$(p_1', p_1'', \cdots, p_1^{(n-1)}; p_2', p_2'', \cdots, p_2^{(n-1)}; \cdots; p_k', p_k'', \cdots, p_k^{(n-1)}; \cdots)$$

obtained from the recursion formulae

$$u_k' = u_{k+1}^{(n)}, \quad u_k^{(i+1)} = u_{k+1}^{(i)} + p_k^{(i)} u_{k+1}^{(n)}, \quad i = 1, 2, \cdots, n-1,$$

where the $p_k^{(i)}$ are chosen by some law. Convergents are defined as in the Jacobi theory. The choice of partial quotients is called admissible if (1) p_k' is the largest integer not greater than u_k''/u_k', (2) $p_k^{(i)}$ is an integer such that $u_{k+1}^{(i)}/u_{k+1}' \geqq 0$, (3) $p_k^{(i)} \geqq 0$, $p_k^{(n-1)} \geqq 1$ for $k > 1$. It is proved that if the partial quotients are admissible and bounded, the convergents converge to the given ratios. If an admissible continued fraction becomes periodic, the expansion represents $n-1$ numbers of the same algebraic field. Results, mostly known, are obtained in the theory of units of quadratic and cubic fields. The theory of matrices with integral elements is used effectively throughout.

C. C. MacDuffee (Madison, Wis.).

Calculus

★Rose, Clarence E. Matrix and Tensor Algebra. Chemical Publishing Co., Inc., New York, N. Y., 1940. viii+143 pp. $4.00.

★Cohen, Abraham. Elements of Calculus. D. C. Heath and Company, Boston, Mass., 1940. v+583 pp. $3.50.

★Franklin, Philip. A Treatise on Advanced Calculus. John Wiley & Sons, Inc., New York, 1940. xiv+595 pp. $6.00.

The book begins with the fundamental notions of real numbers, limits, continuity and convergence. The author then develops the logarithmic, exponential and trigonometric functions from their functional equations. Differential and integral calculus is developed from the beginning, but from an advanced standpoint. A chapter on the elementary functions of a complex variable precedes the first chapter on integration. Besides the usual topics common to most books on advanced calculus, several other less commonly included subjects are treated, such as the measure and content of sets, Stieltjes integrals, equi-continuity, the theory of functions of several complex variables, existence theorems for differential equations, Fourier integrals and Laplace transforms, Bernoulli polynomials and the Euler-Maclaurin sum formula, etc. Some topics, such as Bessel's functions, appear only among the varied exercises at the ends of the chapters. *J. S. Frame* (Providence, R. I.).

★v. Kármán, Theodore and Biot, Maurice A. Mathematical Methods in Engineering. An introduction to the mathematical treatment of engineering problems. McGraw-Hill Book Co., Inc., New York, 1940. xii+505 pp. $4.00.

This book aims to increase the mathematical power of the student who has completed a first course in calculus by presenting the mathematical approach to certain representative problems, principally from the field of mechanical engineering. The topics treated include differential equations, Bessel functions, dynamics, oscillations, vibration and stability problems, the operational calculus and simple difference equations. The pithy chapter on dynamics, which includes the theory of the gyroscope and Lagrange's generalized coordinates, is outstanding. The mathematics teacher may regret the omission of the motivation for certain mathematical setups, and the small number of problems. However, there are adequate references to fuller treatments and the problems given seem well selected. While their emphasis is on applications rather than on theory, the authors have used commendable care in their mathematical statements so that the results stated without proof are not only intelligible to an immature reader, but also reasonably accurate. The book is a well above average contribution to the needs of engineering students. *P. Franklin.*

Stoïlow, S. Sur l'inversion des transformations dont le déterminant fonctionnel s'annule sans changer de signe. Bull. Math. Phys. Éc. Polytéch. Bucarest 10, 1–3 (1940). [MF 2870]

If the Jacobian $\partial(P, Q)/\partial(x, y) = J$ is not zero in a neighborhood of a point, then the transformation $u = P(x, y)$, $v = Q(x, y)$ is known to be one-to-one in that neighborhood. The author proves that this is still true with the

possible exception of certain isolated points, if $J \geqq 0$, and certain additional assumptions are satisfied.

O. Szász (Cincinnati, Ohio).

Fubini, Guido. The mean-value theorem for non-differentiable functions. Publ. Inst. Mat. Univ. Nac. Litoral 2, 25–28 (1940). (Spanish) [MF 2512]

Levi, Beppo. On a theorem of Weierstrass, Rolle's theorem and the above theorem of Fubini. Publ. Inst. Mat. Univ. Nac. Litoral 2, 29–34 (1940). (Spanish) [MF 2513]

1. Let $\phi(x)$ be a continuous function, $a \leqq x \leqq a+k$. By means of the primitive function $\int \phi(x)dx$ the author shows the following. For each h $(0<h<k)$ there is a corresponding x_h $(a \leqq x_h < x_h + h \leqq k)$ such that

$$\lim_{h\to 0} \frac{\phi(x_h+h)-\phi(x_h)}{h} = \frac{\phi(a+k)-\phi(a)}{k}.$$

This is then proved anew without primitive functions by an argument showing that we may also assume that the limit $\lim_{h\to 0} x_h$ exists.

2. A discussion showing how the various entities whose existence is stated by the theorems named in the title may be effectively constructed, at least theoretically.

I. J. Schoenberg (Waterville, Me.).

Kimball, W. S. Partial derivatives of derivatives. Philos. Mag. (7) 30, 190–222 (1940). [MF 3030]

This paper appears to be concerned principally with the introduction of a "new definition for a derivative" of y with respect to x, which turns out to be the quotient of the differential of y, which corresponds to a unit increment of x, by this unit increment. It is not at all clear where the novelty of this concept lies, nor is it apparent what is to be gained from the definition in spite of the statement that we "are obliged to leave by the way-side the mathematician's definition of a derivative which has fallen down, and proceed according to the physicist's, on which rests the superstructure of applied science, and whose position is thus impregnable." That the choice of units in which variable magnitudes are measured is to be kept in mind when a rate of change is calculated by means of a derivative should be a commonplace; it certainly does not call for the extended discussion which the author devotes to it.

A. Dresden (Swarthmore, Pa.).

Müller, Max. Über die Vertauschbarkeit von Grenzübergang und Differentiation. Jber. Deutsch. Math. Verein. 50, 93–104 (1940). [MF 3068]

The author offers a simplified treatment for the formulation of a criterion that the limit function of a sequence of functions of one variable be differentiable at a given point. Many writers content themselves with stating conditions which while sufficient are obviously not necessary. The author introduces the term "unigradient" ("gleichgradig") and expresses several of his theorems in terms of this concept. The functions of a sequence $f_1(x), f_2(x), \cdots, f_n(x), \cdots$ are unigradiently differentiable at a place x_0 if the derivatives $f_n'(x_0)$ exist and if furthermore, for each positive number ϵ, there is a $\delta(\epsilon, x_0)$, independent of n, such that for each h, where $0<|h|<\delta$, it is true that $|f_n(x_0+h)-f_n(x_0) - hf_n'(x_0)| < \epsilon|h|$. The fundamental theorem shown to be equivalent to Dini's criterion is the following: If the functions $f_1(x), f_2(x), \cdots$ converge to the function $f(x)$ at a place x_0 in the interval $a \leqq x \leqq b$, and are unigradiently differentiable at x_0, then $f'(x_0)$ also exists and is the limit (with respect to increasing n) of $f_n'(x_0)$.

A. A. Bennett.

Menger, Karl. On Green's formula. Proc. Nat. Acad. Sci. U. S. A. 26, 660–664 (1940). [MF 3146]

By taking the limit of a certain identity the author proves Green's formula $\iint_R u_y dxdy = -\int_R u(x, y)dx$ for a rectangle R under weaker assumptions than the usual ones in which the proof is done by iterated integration. For more complicated domains the formula is derived by use of elementary continuity properties of the integrals.

W. T. Martin.

Vicente Gonçalves, J. Sur l'intégrale prise sur un contour variable. Portugaliae Math. 1, 343–345 (1940). [MF 2895]

It is proved quite easily that under certain assumptions $\int_{C_n} f(z)dz \to \int_C f(z)dz$ as $C_n \to C$.

O. Szász.

Vescan, T. An example of quantification of classical mechanics and the movement on a cardoid. Bol. Mat. 13, 228–233 (1940). (Spanish) [MF 3151]

Brusotti, Luigi. Sul luogo dei contatti fra coniche di un sistema algebrico ∞^1 dotato di due punti-base. Boll. Un. Mat. Ital. (2) 2, 200–205 (1940). [MF 2973]

Kàrteszi, Francesco. Sopra un sistema di coniche ∞^1 e di indice 4. Boll. Un. Mat. Ital. (2) 2, 314–320 (1940). [MF 2982]

The author considers a system of conics defined by an equation containing one parameter to degree 4, and imposes the condition that each conic of the system shall have four point contact with a given fixed conic and also touch a given fixed line. A related problem was considered by Berzolari [Boll. Un. Mat. Ital. (2) 2, 1–10 (1939); cf. these Rev. 1, 300] and by Brusotti [cf. the preceding title] by different methods. In particular, those points are determined at which the hyperosculating conics at different points touch each other. The method employed is parametric representation, as used in analytic projective geometry.

V. Snyder.

Maeda, Jusaku. On systems of rectangular hyperboloids and orthogonal hyperboloids associated with a point of a surface. Tôhoku Math. J. 47, 24–34 (1940). [MF 2620]

The author computes in an elementary way various loci connected with sets of quadrics which have a contact of the second order with a fixed one at a given point. Of course, the fixed quadric can be replaced by any surface which has a contact of the second order with it in the given point.

P. Scherk (New Haven, Conn.).

Theory of Functions of Complex Variables

∗Phillips, E. G. Functions of a Complex Variable with Applications. Oliver and Boyd, Edinburgh; Interscience Publishers, Inc., New York, 1940. xi+140 pp. \$1.50.

A concise elementary account of the fundamental ideas and theorems including two chapters devoted to conformal representation (general theory and special transformations), a chapter on the complex integral calculus and one on the calculus of residues. There are numerous examples.

Curtiss, J. H. On extending the definition of a harmonic function. Amer. Math. Monthly **47**, 225–228 (1940). [MF 1927]

Commenting upon a statement of J. Beek, Jr. [Amer. Math. Monthly **46**, 587–588 (1939)] that a conjugate function of a harmonic function $U(x, y)$, defined for complex x, y, may be found by forming the imaginary part of

$$2U\left(\frac{z+c}{2}, \frac{z-c}{2i}\right), \quad z=x+iy, c=\text{constant},$$

the author points out that it is not correct unless a suitable definition of $U(x, y)$ for complex x and y is employed. Let $V(x, y)$ be a conjugate function of $U(x, y)$ and let $f(z)=U(x, y)+iV(x, y)$, $z=x+iy$, z in a simply connected region R. The author shows that any sequence of harmonic polynomials $P_n(x, y)$ converging uniformly to $U(x, y)$ in any closed sub-region of R has the property that for complex z_1, z_2, $P_n(z_1, z_2)$ converges uniformly to $\frac{1}{2}\{f(z')+\overline{f(\bar{z}'')}\}$, where $z'=z_1+iz_2$, $z''=z_1-iz_2$ and z_1 and z_2 are so chosen that z' and $\bar{z}''$ are in R. $U(z_1, z_2)=\frac{1}{2}\{f(z')+\overline{f(\bar{z}'')}\}$ is then a definition of $U(x, y)$ for complex x, y for which Beek's statement holds. *S. E. Warschawski* (St. Louis, Mo.).

Ghika, Alexandre. Sur une inégalité que vérifient les fonctions de carré représentable par l'intégrale de Cauchy. C. R. Acad. Sci. Paris **210**, 598–600 (1940). [MF 3034]

The author establishes the following result: D is a region of the x-plane bounded by a closed rectifiable Jordan curve C. The function $f(x)$ is analytic for $x\epsilon D$ and has the property that $\lim_{x\to z} f(x)(z\epsilon C)$ exists for almost all z on C, where $\lim_{x\to z} f(x)$ is defined on paths in D which are not tangential to C. Furthermore, $\lim_{x\to z} f(x)$ is to coincide for almost all z with $f(z)(\epsilon L^2$ on $C)$. Such functions $f(x)$ are said to belong to the class $\Omega(C)$. Now let C_1 denote a closed rectifiable curve contained in $D+C$. Under these hypotheses for $f\epsilon\Omega(C)$

$$\int_{C_1} |f(z)|^2 ds \leqq M^2 \int_C |f(z)|^2 ds,$$

where M is independent of $f(z)$. The proof is based on an inequality of Toeplitz and Hellinger and the expansion of $f(z)$ in terms of functions normal and orthogonal on C and C_1, respectively, which are obtained from the set

$$\left[(z-a)^m, \frac{1}{(z-a)^{n+1}}\right], \quad m, n=0, 1, 2, \cdots,$$

where a is interior to both C and C_1.

The inequality of the author is closely related to a special case of an inequality due to R. M. Gabriel [Proc. London Math. Soc. (2) **28**, 121–127 (1928)]. The hypotheses of the present result are less restrictive, but the inequality is not as precise as that of Gabriel. *M. H. Heins.*

Spencer, D. C. On an inequality of Grunsky. Proc. Nat. Acad. Sci. U. S. A. **26**, 616–621 (1940). [MF 2896]

Let W be a Riemann domain, covering the point w $n(w)$ times, and write

$$p(R)=\frac{1}{2\pi}\int_{-\pi}^{\pi} n(Re^{i\varphi})d\varphi.$$

Suppose $n_0=n(0)>0$. The author proves that, if

$$\int_0^{R_1} p(R)d(\pi R^2) \leqq n_0\pi R_1^2 \qquad \text{for all } R_1>0,$$

then

$$\text{(S)}\quad \exp\left\{\frac{1}{n_0}\int_0^\infty \log R^2 d[-p(R)]\right\} \leqq \frac{1}{n_0}\int_0^\infty R^2 d[-p(R)],$$

with equality if and only if either the left side is infinite or $p(R)=n_0$ for $0\leqq R<R_0<\infty$ and $p(R)=0$ elsewhere. (S) can be expressed in terms of integrals around the boundary of W, and in this form has been proved by Grunsky [Schr. Math. Sem. u. Inst. Angew. Math. Univ. Berlin **1**, 95–140 (1932); in particular p. 114] under the more restrictive hypothesis $n(w)\leqq n_0$. Grunsky, observing that (S) is a case of the inequality of the arithmetic and geometric means, but with weight function not necessarily positive, reduced it to an algebraic inequality. The author's proof uses simple "real variable" methods and is considerably shorter. As an application, the author sharpens a generalization of Schwarz's lemma given by Bermant [C. R. Acad. Sci. Paris **207**, 31–33 (1938)], extending earlier results obtained independently by Golusin [Rec. Math. [Mat. Sbornik] N.S. **2** (**44**), 617–619 (1937); the author's reference is incorrect] and Grunsky [Jber. Deutsch. Math. Verein. **48**, 51–52 (1938)]. The author's result is that, if $f(z)$ maps $|z|<1$ on a Riemann domain W, $f(0)=0$, W^* is the star of W with respect to $w=0$, and $Z^*=f^{-1}(W^*)$, then

$$\pi^2|f'(0)|^2 \leqq (\text{Area } Z^*)(\text{Area } W^*),$$

with equality only if $f(z)=cz$. The author points out that this result could be obtained by using (S) in Bermant's proof; however, he gives an independent proof based on (S) and a lemma of his own on conformal mapping.

R. P. Boas, Jr. (Durham, N. C.).

Stoïlow, S. Sur une extension topologique du principe du maximum du module et ses applications à la théorie des fonctions. Acad. Roum. Bull. Sect. Sci. **23**, 28–30 (1940). [MF 3086]

The author establishes the following topological extension of the principle of the maximum. Let X and Y be topological spaces, and $Y=T(X)$ a continuous transformation of X on Y, which carries every open set of X into an open set of Y (interior transformation in the wide sense). Let F be a closed and compact set in X, G the set of interior points of F, and $H=F-G$ the frontier of F. It is shown that, if y_1 and y_2 are both contained in a continuum $K\subset Y$, such that $K\cdot T(H)=0$, then y_1 and y_2 are either both contained in $T(G)$ or both in the complement of $T(G)$. Let E be a Euclidean space, $\mathcal{E}=T(E)$ an interior transformation of E on another Euclidean space $\mathcal{E}$, $\{D_i\}$ a sequence of closed and bounded domains in E, and H_i the frontier of D_i. It follows from the above theorem that, if $\lim_{i\to\infty} T(H_i)=\infty$, then either $\lim_{i\to\infty} T(D_i)=\infty$ or $\sum_i T(D_i)=\mathcal{E}$.

As an application, the author considers the function

$$w=\varphi(z)=\sum_{n=0}^{\infty} a^n z^{n!/a^{2n}},$$

where a is an integer greater than 5; this function is analytic for $|z|<1$. The above result together with known properties of $\varphi(z)$, established by Lusin and Privaloff, imply that for this function every ray issuing from the origin is a line of Julia. *E. F. Beckenbach* (Ann Arbor, Mich.).

Robinson, Raphael M. On the mean values of an analytic function. Bull. Amer. Math. Soc. **46**, 849–851 (1940). [MF 2933]

Let $f(z)$ be regular for $|z|<1$, and the mean $M_p(r)$ defined by $2\pi\{M_p(r)\}^p=\int_{-\pi}^{+\pi}|f(re^{i\theta})|^p d\theta$. The paper settles two

problems raised and discussed by Nehari [C. R. Acad. Sci. Paris **206**, 1943 (1938); **208**, 1785 (1939)]. (a) Let $M_1(r)\leqq 1$, $r<1$; then ($p>1$ an integer) $M_p(r)\leqq 1$ for $r\leqq p^{-\frac{1}{2}}$ and this bound $p^{-\frac{1}{2}}$ cannot be replaced by a larger one. (b) Let $\int_0^1|f(re^{i\theta})|dr\leqq 1$, $-\pi\leqq\theta<+\pi$; then $M_2(r)\leqq 1$ for $r\leqq\frac{1}{2}$ and this bound $\frac{1}{2}$ cannot be replaced by a larger one.

G. Szegö (Stanford University, Calif.).

Montel, Paul. L'itération. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Revista (2) **3**, 201–211 (1940). [MF 3010]

In this expository article Montel presents a bird's-eye view of iteration theory. Beginning with the simplest examples (e.g., successive rotations of constant angle) he leads up to the important developments of recent years on the iteration of rational functions, with mention of extensions (e.g., to meromorphic functions) and of the relation of the theory to the Abel and Schröder functional equations and to Fuchsian functions. In keeping with the informality of its style, there is no formal statement of theorems in the article, and no proofs. *I. M. Sheffer.*

Keldych, M. Sur l'approximation en moyenne quadratique des fonctions analytiques. Rec. Math. [Mat. Sbornik] N.S. **5** (47), 391–401 (1939). (French. Russian summary) [MF 2308]

Let D be a simply connected region in the z-plane which contains $z=0$. Let $\Pi_n(z)$ denote the Bieberbach polynomial of degree n, that is, the polynomial for which the integral

$$\iint_{(D)}|P_n'(z)|^2dxdy$$

formed with all polynomials $P_n(z)$, $P_n(0)=0$, $P_n'(0)=1$, of degree n, is a minimum. Bieberbach proved [Rend. Circ. Mat. Palermo **38**, 98 (1914)] that the sequence $\{\Pi_n(z)\}$ converges uniformly in any closed sub-region of D to the function $w=f(z)$ ($f(0)=0$, $f'(0)=1$) which maps D conformally onto a circle $|w|<\rho$, provided the boundary Γ of D is of the "Carathéodory class," that is, Γ is also the boundary of another region containing $z=\infty$. In this connection the author establishes the following theorem: If Γ is a Jordan curve with bounded curvature, then for every $\epsilon>0$, $\epsilon<1$, there exists a constant $C(\epsilon)$ depending upon ϵ and D only such that

$$|f(z)-\Pi_n(z)|<\frac{C(\epsilon)}{n^{1-\epsilon}}, \qquad z\epsilon(D+\Gamma).$$

Moreover, he constructs a region D whose boundary curve Γ is analytic except for one point P such that the sequence $\{\Pi_n(z)\}$ diverges at P, thus showing that it is not sufficient to assume Γ a closed Jordan curve to insure uniform convergence of the $\Pi_n(z)$ in $D+\Gamma$. The author calls the system of all polynomials closed in D if for every $F(z)$, regular in D, for which

$$\iint_{(D)}|F(z)|^2dxdy$$

exists, the greatest lower bound of the integral

$$\iint_{(D)}|F(z)-P(z)|^2dxdy$$

is 0, $P(z)$ being an arbitrary polynomial. For a region D of the Carathéodory class the system of all polynomials is closed in D [see Farrel, Bull. Amer. Math. Soc. **40**, 908–914 (1934)]. The author constructs examples of regions, not of Carathéodory class, from which he infers that the class of all regions in which the system of all polynomials is closed cannot be characterized by topological properties but that this class depends on metrical properties of these regions.

S. E. Warschawski (St. Louis, Mo.).

Walsh, J. L. and Sewell, W. E. Sufficient conditions for various degrees of approximation by polynomials. Duke Math. J. **6**, 658–705 (1940). [MF 2722]

The present paper is a contribution to a class of problems designated by the authors as "Problem β." The following is a typical special case. Let C be a Jordan curve, C_ρ, $\rho>1$, the level curve in the exterior of C defined by means of Green's function in the customary way. Let $f(z)$ be analytic in the interior of C_ρ, and $f^{(p)}(z)$ continuous in the closed interior of C_ρ, and let $f^{(p)}(z)$ satisfy a Lipschitz condition of exponent α $(0<\alpha\leqq 1)$ on C_ρ. Then polynomials $p_n(z)=az^n+\cdots$ exist such that for z on C

$$|f(z)-p_n(z)|<Mn^{-p-\alpha}\rho^{-n}.$$

This is true in particular for polynomials $p_n(z)$ defined by interpolatory conditions on the roots of certain well-known polynomials associated with the curve C (Tchebycheff, Faber and orthogonal polynomials). Extension of these results to a set of Jordan curves and to more general sets is possible. Approximations of Bessel type are also considered. The special cases of a segment and of a lemniscate are particularly investigated. *G. Szegö.*

Lammel, Ernst. Über Approximation im Einheitskreise regulärer Funktionen eines komplexen Argumentes. Monatsh. Math. Phys. **49**, 199–208 (1940). [MF 2996]

Let $\{a_\mu\}$ be a sequence of points belonging to the region $|z|\leqq\rho<1$ and let $\{b_\mu\}$ be a finitely situated sequence of points belonging to the region $|z|\geqq 1$. If $f(z)$ is an analytic function regular in $|z|<1$, then there is a unique sequence of rational functions $S_n(z)$, the partial sums of

$$C_0+\sum_{\nu=1}^{\infty}C_\nu\prod_{\mu=1}^{\nu}\frac{z-a_\mu}{z-b_\mu}, \tag{1}$$

satisfying the conditions that, when a_λ occurs κ-times in the sequence $\{a_\mu\}$, then $S_n^{(p)}(a_\lambda)=f^{(p)}(a_\lambda)$, $0\leqq p\leqq\kappa-1$; $f^{(0)}(z)=f(z)$. The author obtains necessary and sufficient conditions for relations between $\{a_\mu\}$ and $\{b_\mu\}$, namely,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\mu=1}^{n}\left(\bar{a}_\mu{}^k-\frac{1}{b_\mu}\right)=0, \qquad k=1,2\cdots, \tag{2}$$

in order for the series (1) of rational functions to converge to $f(z)$ for each z in $|z|<1$. *A. Gelbart* (Raleigh, N. C.).

Boas, R. P., Jr. Expansions of analytic functions. Trans. Amer. Math. Soc. **48**, 467–487 (1940). [MF 3168]

The problem is that of expanding analytic functions in generalized Taylor's series of the form $f(z)=\sum c_ng_n(z)$, where the $g_n(z)$ are in some sense "nearly" the same as z^n. In particular, $g_n(z)$ may be of the form $z^n[1+h_n(z)]$, where the $h_n(z)$ are analytic and uniformly bounded for $z<r$ and vanish at $z=0$. In this case it is known that any function $f(z)$ which is analytic for $z<r$ has such an expansion for some circle $z\leqq s<r$. Various estimates for s have been given in the literature, most of them based on the substitution of power series for $h_n(z)$ and rearrangement of the resulting double series. The present paper presents an entirely new

method of attack on the problem by considering the spaces $L_p(r)$ of functions analytic for $z<r$ and having a uniformly bounded L^p norm on all circles $z=\rho<r$. To these spaces is applied a theorem of Paley and Wiener which gives a criterion of "nearness" such that in a complete normed linear space every sequence of elements sufficiently "near" a given base will also be a base. The simplest result is that s may be obtained from the inequality $h(s)<1$, where $h(s)$ is a majorant for the $h_n(z)$, that is, the coefficients in its power series dominate the corresponding coefficients in the power series for $h_n(z)$. This includes and goes beyond most of the previously given estimates for s. In particular, for the case $1+h_n(z)=e^{\alpha_n z}$, where $|\alpha_n|\leqq 1$, there is obtained the value $s=\log 2=0.693$, whereas the best previously published estimate was $1/e$. [The reviewer gets 2/3 by a direct application of an earlier theorem.] These expansions are applied to the study of the derivatives of an entire function of exponential type k. A lower bound is thus obtained for the rate at which $|f^n(\alpha_n)|^2$ may change with n as n tends to infinity, provided $k<\log 2$ and $f(0)\neq 0$. A parallel theory is given for the uniform approximation to analytic functions by sums of a similar form. Conditions are also given for the stronger case where the expansion holds for $z\leqq r$.

P. W. Ketchum (Urbana, Ill.).

Broggi, Ugo. Su due teoremi di E. Landau. Boll. Un. Mat. Ital. (2) 2, 112–114 (1940). [MF 2969]

The author proves, as an easy corollary of the Vivanti-Pringsheim theorem for power series, the following theorem. Let the function $f(s)$ be regular in the half plane $\Re(s)>\alpha-\epsilon$, ϵ being a positive, sufficiently small number. Let $f(s)$ satisfy the conditions $(-1)^r f^{(r)}(\alpha_n)\geqq 0$, $r=1, 2, \cdots$, with $\alpha<\alpha_1<\alpha_2<\cdots$, $\lim \alpha_n=\infty$. Let ρ be the radius of convergence of the Taylor series of $f(s)$ about the point α. Then $f(s)$ is regular in the half plane $\Re(s)>\alpha-\rho$ and $\alpha-\rho$ is a singular point. He then uses this result to prove the following theorem. Let μ be the abscissa of convergence of the Laplace integral

$$f(s)=\int_0^\infty e^{-ts}\varphi(t)dt.$$

Let $\varphi(t)\geqq 0$ almost everywhere for $t>0$. Then μ is a singular point. [It may be remarked that this theorem is, strictly speaking, not new; the known proof of it, which assumes $\varphi(t)\geqq 0$ everywhere, is indeed valid without any change in the present case if only the integral is taken in the sense of Lebesgue.] From this theorem the author deduces the Landau analogue, for factorial series, of the Vivanti-Pringsheim theorem for power series. He finally proves, as a particular case of the first theorem, the analogue (also due to Landau) for Dirichlet series of the Vivanti-Pringsheim theorem for power series.

A. González Domínguez (Buenos Aires).

Wolf, František. Ein Eindeutigkeitssatz für analytische Funktionen. Math. Ann. 117, 383 (1940). [MF 3002]

Suppose that $f(z)$ is analytic in $|z|<1$, and that $\varliminf_{r\to 1} |f(re^{i\theta})|<\infty$ in a set $M(\theta)$ which is the product of infinitely many open sets (therefore of Baire's class 1) and everywhere dense in an interval (α, β). Then if $\varliminf_{r\to 1} |f(re^{i\theta})|=0$ almost everywhere in (α, β), the author shows that $f\equiv 0$. The proof is only a few lines in length; it is shown that there exists an interval $(\gamma, \delta)\subset(\alpha, \beta)$ such that $|f(re^{i\theta})|$ is uniformly bounded from above if $0\leqq r<1$, $\gamma\leqq\theta\leqq\delta$, and the result then follows from known theorems. Line 12, in which there is a typographical error, should read:

$$\mathfrak{A}(N;\theta)=\lim_{\rho\to 1}\mathfrak{A}(\rho, N;\theta).$$

D. C. Spencer (Cambridge, Mass.).

Lambin, N. Sur une classe de points singuliers essentiels. C. R. (Doklady) Acad. Sci. URSS (N.S.) 25, 467–469 (1939). [MF 2075]

The author announces several results (without proofs) regarding analytic functions $f(z)$ with an essential singularity at $z=z_0$ which do not vanish and possess a non-vanishing derivative in a neighborhood of z_0. One of these results deals with a decomposition of any (sufficiently small) neighborhood of z_0 by means of the lines $\arg f(z)=\text{const.}$ into so-called "characteristic regions" which have the property that the asymptotic values of $f(z)$ may be obtained when $z\to z_0$ along the contours of these regions. Further statements concern the form of $f(z)$ in a neighborhood of z_0 as depending upon the asymptotic values and the paths along which $f(z)$ approaches them. An example of these results that can be stated briefly is: If 0 and ∞ are the only asymptotic values of $f(z)$ at z_0, then $f(z)=(z-z_0)^k e^{p(z)}$, where $p(z)$ has a pole at $z=z_0$ and k is a non-negative integer.

S. E. Warschawski (St. Louis, Mo.).

Noshiro, Kiyoshi. On the singularities of analytic functions. Jap. J. Math. 17, 37–96 (1940). [MF 3156]

The main purpose of the writer is to prove the following theorem which generalizes a known theorem of Iversen [Thèse, Helsingfors, 1914]. Denote by $\omega=f(z)$ an analytic function, by $z=\phi(\omega)$ its inverse, by Φ the Riemann's covering surface associated with ϕ [Stoïlow, Leçons sur les principes topologiques de la théorie des fonctions analytiques, Paris, 1938; Ahlfors, C. R. Congrès Intern. Math., Oslo, 1936]; $\phi(q)$ is uniform and meromorphic on Φ. Any transcendental singularity Ω, with coordinate $z=\omega$, of $f(z)$ corresponds to an asymptotic path Γ, on Φ, of $\phi(q)(q\epsilon\Phi)$, ω being the asymptotic value along Γ. The converse is obvious. A new proof of Beurling's theorem on the cluster sets [Thèse, Uppsala, 1933] and some applications of this theorem are given (that is, if $f(z)$ is uniform and meromorphic in D, $z_0\epsilon\bar{D}-D$, the cluster set $S_{z_0}^{(D)}$ is the set of all the values a such that $f(z_\nu)\to a$ when $z_\nu\to z_0$, $z_\nu\epsilon D$).

S. Mandelbrojt (Houston, Tex.).

Iyengar, K. S. K. A property of integral functions with real roots and of order less than two. Proc. Indian Acad. Sci., Sect. A. 12, 223–229 (1940). [MF 2952]

The author deduces two inequalities found by Erdös and Grünwald [Ann. of Math. (2) 40, 537–548 (1939); these Rev. 1, 1] from more precise results. These are stated in four theorems of which we quote here the first (T_1) in slightly modified form: Assume that $\phi(x)$ is either a real polynomial whose roots are all real, or an entire function that is a limit of such polynomials. Assume further that $\phi(a_1)=\phi(a_2)=0$, $\phi(x)>0$ in $a_1<x<a_2$, $\phi'(a_1)=m_1>0$, $\phi'(a_2)=-m_2<0$. Then, in $a_1<x<a_2$,

$$\phi(x)\geqq (a_2-x)(x-a_1)\frac{(m_1m_2)^{\frac{1}{2}}}{a_2-a_1}\exp\left[\frac{\lg m_2-\lg m_1}{a_2-a_1}\left(x-\frac{a_1+a_2}{2}\right)\right].$$

This statement is slightly more general than that of the author [the entire function $\phi(x)$ considered by the author may be multiplied by $e^{-\gamma x^2}$, where $\gamma\geqq 0$; see Laguerre,

Oeuvres, Paris, 1898, vol. 1, pp. 174–177 and Rend. Circ. Mat. Palermo **36**, 279–295 (1913)]. The proof is identical with the author's simple proof: the difference of the logarithms of the functions on both sides of the inequality is represented by a convex curve in $a_1<x<a_2$.

G. Pólya (Providence, R. I.).

Keldych, M. et Lavrentieff, M. Sur un problème de M. Carleman. C. R. (Doklady) Acad. Sci. URSS (N.S.) **23**, 746–748 (1939). [MF 1954]

Let E be a continuum in the plane of the complex variable z, containing $z=\infty$. E is called a "continuum of Carleman" if for any function $f(z)$, continuous on E ($|z|<\infty$), and for any positive, continuous function $\epsilon(r)$, $r\geqq 0$, there exists an integral function $F(z)$ such that on E: $|f(z)-F(z)|<\epsilon(|z|)$. T. Carleman [Ark. Mat. Astr. Fys. **20** B, no. 4, 1927] and R. Roth [Comment. Math. Helv. **11**, 77–125 (1938)] have given examples of such continua. In the present paper it is proved that in order that E be a "continuum of Carleman" it is necessary and sufficient that (1) E be nowhere dense and that (2) there exist an increasing function $r(t)$, $\lim_{t\to\infty} r(t)=\infty$, such that for every z_1 not on E there exists a Jordan curve joining z_1 with the point at infinity which lies outside E and outside the circle $|z|<r(|z_1|)$.

S. E. Warschawski (St. Louis, Mo.).

Boas, R. P., Jr. Univalent derivatives of entire functions. Duke Math. J. **6**, 719–721 (1940). [MF 2724]

Let $M(r)$ denote the maximum of $|f(z)|$ in $|z|\leqq r$. If $f(z)$ is an entire function for which

$$\limsup_{r\to\infty} \frac{1}{r} \log M(r)<\log 2,$$

and if $f(z)$ is not a polynomial, an infinite number of the derivatives of $f(z)$ are univalent in the unit circle $|z|\leqq 1$. It is not known if the constant log 2 in this theorem can be replaced by a larger one. If $f(z)$ is an entire function, not a polynomial, of order less than one, or of order one and minimum type, then corresponding to any increasing sequence of numbers r_n there is an increasing sequence of integers k_n such that $f^{(k_n)}(z)$ is univalent in $|z|<r_n$ ($n=1, 2, \cdots$). Assuming that neither $f(z)$ nor any derivative is univalent in the unit circle, the author shows that $f(z)$ must be a constant. To this end he makes use of representation for $f(z)$ obtained by G. Pólya [Math. Z. **29**, 549–640 (1929)]:

$$f(w)=\int_C e^{zw}F(z)dz,$$

where $F(z)$ is analytic on C, and substitutes an expansion for e^{zw} of the form

$$e^{zw}=1+\sum_{n=1}^{\infty} c_n(w)z^{n-1}\cdot\frac{e^{a_n z}-e^{b_n z}}{a_n-b_n},$$

where

$$|a_n|\leqq 1,\quad |b_n|\leqq 1,\quad a_n\neq b_n,\quad f^{(n-1)}(a_n)=f^{(n-1)}(b_n),\qquad n=1, 2, \cdots.$$

On account of the last equality integration term by term obtains $f(w)$ as a constant. If $f(z)$ has only a finite number of univalent derivatives the argument above may be applied to $f^{(k+1)}(z)$ which results in $f(z)$ being a polynomial.

M. S. Robertson (New Brunswick, N. J.).

Spencer, D. C. Note on some function-theoretic identities. J. London Math. Soc. **15**, 84–86 (1940). [MF 2516]

Let $f(z)=Re^{i\Phi}$ be regular in $|z|<1$. For $0<r<1$ the author defines $n(r, w)$ to be the number of times that $f(z)$ takes the value w in $|z|<r$, and $p(r, R)$ by the relation

$$p(r, R)=\frac{1}{2\pi}\int_{-\pi}^{\pi} n(r, Re^{i\psi})d\psi.$$

Set

$$A=\int_{|z|=r} R^\lambda d\Phi,\quad B=\int_{R=0}^{\infty} p(r, R)d(\pi R^\lambda),$$

$$C=r\frac{d}{dr}\int_{-\pi}^{\pi}|f(re^{i\theta})|^\lambda d\theta,$$

$$D=\lambda\int_0^r\int_{-\pi}^{\pi}|f(\rho e^{i\theta})|^{\lambda-2}|f'(\rho e^{i\theta})|^2\rho d\rho d\theta,$$

where $0<r<1$, $\lambda>0$. The author proves by "geometrically intuitive" arguments the identities $\frac{1}{2}A=B$, $2B=D$; $\lambda D=C$ is the well-known Hardy-Stein identity, and $C=\lambda A$ is an identity of Prawitz [Ark. Mat. Astr. Fys. **20** A, no. 6 (1927)]. Since any three of these identities imply the fourth, a new proof of the Hardy-Stein identity is obtained. [The author's reference to Hardy should be Proc. London Math. Soc. (2) **14**, 269–277 (1915)].

R. P. Boas, Jr.

Spencer, D. C. On finitely mean valent functions. II. Trans. Amer. Math. Soc. **48**, 418–435 (1940). [MF 3165]

[This paper is a sequel to one of the same title appearing in the Proc. London Math. Soc.] If $f(z)$, regular for $|z|<1$, takes no value w more than p times in the unit circle, we say $f(z)$ has valency p. If $W(R)$ is the area of that portion of the transform of $|z|<1$ by f which lies in $|w|\leqq R$ (regions covered multiply being counted multiply) and if $W(R)\leqq p\pi R^2$ for all $R>0$, p being any positive number, we say $f(z)$ is p mean valent (p-m.v.). Many of the known theorems for p-valent functions are extended in the first paper to p-m.v. functions. In this second paper, using the distortion theory of Ahlfors, which Cartwright had already employed for p-valent functions, the author obtains inequalities for the rate of growth of p-m.v. functions and for the coefficients of these functions analogous to known ones for p-valent functions. In particular, if

$$f_k(z)=a_1z+a_{k+1}z^{k+1}+a_{2k+1}z^{2k+1}+\cdots$$

is p-m.v. and if

$$\mu_{[p/k]}=\max\left[|a_1|, \cdots, |a_{[p/k]}|\right],$$

then

$$\frac{1}{2\pi}\int_0^{2\pi}|f_k(re^{i\theta})|d\theta\leqq K(p, k)\mu_{[p/k]}(1-r)^{-2p/k},$$
$$|a_n|\leqq K(p, k)\mu_{[p/k]}n^{(2p/k)-1},\qquad p>\tfrac{1}{2}k.$$

An example is given to show that the condition $p>\frac{1}{2}k$ is necessary.

An investigation of the behavior of p-m.v. functions on Jordan paths tending to points on the circumference of the unit circle yields among others the following theorems. Suppose $f(z)$ is p-m.v. and that E_θ is a set of distinct points. If to each point θ of E_θ there corresponds at least one path $P(\theta)$ for which

$$\liminf_{P(\theta)} (1-r)^{\alpha(\theta)}|f(z)|>0,$$

then $\sum_{E_\theta}\alpha(\theta)\leqq 2p$. An example is given to show that the conclusion of this theorem is a best possible one when p is integral. It is also shown that the hypothesis cannot be relaxed to the extent of replacing lim inf by lim sup. A further theorem states that, if $f(z)$ is p-m.v. and if $f(z)=O(1)$ on some path ending at $e^{i\theta_0}$, then on any Jordan path

ending at $e^{i\theta_0}$

$$\limsup (1-r)^{2p}|f(z)|=0.$$

The author shows that this theorem is a best possible one. The full strength of the hypothesis $W(R)\leqq p\pi R^2$ is not necessary for the truth of the theorems and the author gives several weaker conditions. *M. S. Robertson.*

Schaeffer, A. C. and Szegö, G. Polynomials whose real part is bounded on a given curve in the complex plane. Amer. J. Math. 62, 868–876 (1940). [MF 2886]

Continuing the previous work of Szegö [Math. Z. 23, 45–61 (1925)] and using conformal mapping and other tools of the theory of analytic functions, the authors prove the following result: Let Γ be a closed Jordan curve consisting of a finite number of analytic arcs which join in such a way that the exterior angle is always greater than zero and less than 2π. Let $f(z)$ be a polynomial of degree n satisfying $|\Re f(z)|\leqq 1$ on Γ. Then (I) $|f'(z_0)|\leqq An^{\alpha}$ at an arbitrary point z_0 of Γ, and (II) $|f(z_1)-f(z_2)|<B\log n$ for two arbitrary points z_1 and z_2 in the closed interior of Γ and for $n>1$. The positive numbers A and B are constants, A depending on Γ and z_0, B depending only on Γ, and $\alpha\pi$ is the exterior angle of Γ at z_0. It is not possible to improve the order of either of the bounds as $n\to\infty$. *G. Pólya.*

Lavrentieff, M. und Kwesselawa, D. Über einen Ostrowskischen Satz. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] 1, 171–174 (1940). (Russian. German summary) [MF 2021]

The authors state and indicate a proof of the following theorem: Let D be a simply-connected region in the z-plane which contains the circle $|z|\leqq\theta$ ($\frac{1}{2}\leqq\theta<1$) and which is contained in the circle $|z|<1$. Let $f(z)$ map D conformally onto the circle $|w|<1$ and let $f(0)=0$, $f'(0)>0$. Then, for $|z|\leqq\theta$,

$$|f(z)-z|\leqq K(1-\theta)\log\frac{1}{1-\theta},\quad k \text{ an absolute constant.}$$

This result extends a theorem of A. Ostrowski who proved [Jber. Deutsch. Math. Verein. 39, 78–81 (1930)] that, under the above stated hypotheses, for $|z|\leqq\theta\theta_1(\frac{1}{2}<\theta_1<1)$, $|f(z)-z|\leqq(1-\theta)+2(1-\theta)^{\frac{1}{2}\log 1/\theta_1}$. As $\theta_1\to 1$, this result shows only that $|f(z)-z|$ is bounded.

S. E. Warschawski (St. Louis, Mo.).

Wolff, Julius. Théorème sur les domaines invariants dans la représentation conforme. C. R. Acad. Sci. Paris 210, 658–659 (1940). [MF 3041]

Let $z=x+iy$, and suppose that $z_1(z)$ transforms (conformally) the half-plane $x>0$ into a domain D_1 contained in $x>0$. If $n>1$, let $z_n=z_1\{z_{n-1}(z)\}$, and let D_n be the map of $x>0$ by z_n (so that $D_{n-1}\subset D_n$). Write

$$N=\prod_{n=1}^{\infty}D_n,\quad \lambda=\lim_{z\to\infty}\frac{dz_1}{dz}.$$

The author supposes that λ is real, and that $0<\lambda<1$, and proves that for every ϵ, $0<\epsilon<\pi/2$, there is a number R such that N contains the portion of the sector $|\arg z|<(\pi/2)-\epsilon$ for which $|z|>R$. The result is a generalization of a previous theorem of the author [C. R. Acad. Sci. Paris 204, 1101 (1937)]. *D. C. Spencer.*

Unkelbach, Helmut. Über die Randverzerrung bei schlichter konformer Abbildung. Math. Z. 46, 329–336 (1940). [MF 2402]

Let $w=F(z)$ satisfy the conditions of the lemma of Schwarz; let $|F'(0)|=R$ and arc $F'(0)=\Phi$. The author previously has shown [Math. Z. 43, 739–742 (1938)] that the angle-derivative (Winkelderivierte),

$$D=\lim_{\nu\to\infty}\frac{1-f(z_\nu)}{1-z_\nu},\quad |\text{arc }(1-z_\nu)|<\frac{\pi}{2}-\eta,$$

where η is an arbitrary small positive number independent of ν, satisfies

$$D\geqq\frac{2}{1+R},$$

$$\begin{cases} D\geqq 1+\sin|\Phi| & \text{for } 0\leqq|\Phi|\leqq\pi/2,\\ D\geqq 2 & \text{for } \pi/2\leqq|\Phi|\leqq\pi,\end{cases}$$

the inequalities being sharp; that is, the right-hand members are the precise greatest lower bounds of D for given values of R and Φ. It is now shown that under the additional hypothesis that the function $w=F(z)$ gives a univalent map of $|z|<1$, the above inequalities can be replaced by $D\geqq 1/\sqrt{R}$, $D\geqq e^{|\Phi|}$ for $|\Phi|\leqq\pi$, which also are sharp. In the proof use is made of Peschl's approximation theorem, that the functions under discussion are limits of sequences of iterated slit-mapping functions. The author makes essential use of the qualifying hypothesis, which he proposes to discard in a subsequent paper, that the sequence of angle-derivatives of the approximating functions converges to the angle-derivative of $F(z)$. The inequalities are further improved under additional hypotheses that the map is star-shaped, or convex, etc. *E. F. Beckenbach.*

Aumann, Georg. Die Mittelpunktsverzerrung bei konvexen konformen Abbildungen. Math. Z. 46, 80–82 (1940). [MF 1481]

C being a convex domain of the z-plane, the arithmetical mean $m=(z_1+z_2)/2$ of two points z_1 and z_2 in C is evidently also contained in C. Generalizing this operation, the author calls every function $m=F(z_1,z_2)$ of two points z_1, z_2 in a domain $\mathfrak{G}$ an analytic mean in $\mathfrak{G}$ if $F(z_1,z_2)$ is an analytic function of z_1 and z_2, $F(z_1,z_2)=F(z_2,z_1)$, and $F(z,z)=z$. [See G. Aumann, S.-B. Math. Nat. Abt. Bayer. Akad. Wiss. 1934, 48–81; Math. Ann. 111, 713–730 (1935).] By conformal mapping of $\mathfrak{G}$ on a new domain $\mathfrak{G}'$, a new analytic mean $m'=F'(z_1',z_2')$ in $\mathfrak{G}'$ is obtained from $m=F(z_1,z_2)$. If $F(z_1,z_2)$ is an arithmetical mean, $F'(z_1',z_2')$ is called quasi-arithmetical. The principal result of this paper is the determination of all possible values of m for a given domain, two given points z_1, z_2 in it but varying $F(z_1,z_2)$. According to the foregoing remarks it is sufficient to consider as domain $\mathfrak{G}$ the unit circle $|z|<1$ and to subject the points z_1 and z_2 to the condition $z_1+z_2=0$. In this case the result is: $|m|\leqq|z_1|^2$. The author also makes some remarks on analytic mean values in more than 2 variables.

C. Loewner (Louisville, Ky.).

Chufistova, A. M. Approximate conformal mapping with the help of the exponential function. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 6, 119–126 (1939). (Russian) [MF 3296]

The author gives a method for the approximate determination of the function $\zeta=f(z)$ which maps a given simply-connected domain, containing the origin and bounded by a

Jordan curve C, on the unit circle $|\zeta|<1$, with $f(0)=0$, $f'(0)$ real and positive. The function is written in the form $\zeta=ze^{\phi(z)}$, and ϕ is approximated by polynomials

$$\phi_n(z)=\sum_{m=0}^{n-1}a_m z^m$$

which satisfy the condition that $\Re\{\phi_n\}=-\ln|z|$ at a suitably chosen set of points on the contour C. Three examples are given: (i) the mapping of the interior of a square on $|\zeta|<1$; (ii) the interior of the oval of Cassini, $|z^2-1|=$constant, on $|\zeta|<1$; and (iii) the exterior of the ellipse $x^2+by^2=1$ on $|\zeta|>1$. Convergence is discussed.

D. C. Spencer (Cambridge, Mass.).

Biernacki, M. Sur la représentation conforme des domaines étoilés. Mathematica, Cluj 16, 44–49 (1940). [MF 2481]

The following theorems are proved: (1) Let $w=f(z)=a_1z+a_2z^2+\cdots$ map the circle $|z|<1$ onto a region R which is star-shaped with respect to $z=0$, and let $f(z_0)=z_0$, $|z_0|<1$. Consider a system Δ of n straight lines $D_1, D_2, \cdots, D_n$, each of which starts at the origin. Let M_i be the point where D_i first meets the boundary of R. Suppose that $F(x_1, x_2, \cdots, x_n)$ is a continuous, positive and increasing function of each of the variables x_i, $x_i>0$. Then it is possible to rotate Δ about $z=0$ in such a manner that

$$F(OM_1, OM_2, \cdots, OM_n)\geqq F(1, 1, \cdots, 1).$$

(2) Let $w=f(z)$ have the same properties as in (1), and suppose, in addition, that $|z_0|<1/10$ and $|f'(z_0)|=1$. Then the values $w=f(z)$ cover at least a circle of radius

$$\frac{1}{4}\,\frac{(1-|z_0|)^3}{1+|z_0|}$$

about $w=0$. This radius is the best possible. As a corollary of (1) it follows that, if the boundary B of R is represented by the equation $\rho=\rho(\theta)$, $0\leqq\theta\leqq 2\pi$ ($z=\rho e^{i\theta}$, $\rho(\theta)$ integrable), and if $\psi(x)$ is an increasing, continuous and positive function of x, $x>0$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\psi[\rho(\theta)]d\theta\geqq\psi(1).$$

This yields immediately the result that the area of R is not less than π and that the length of B is not less than 2π. As the author points out, (1) is an extension of a result of Bermant [C. R. (Doklady) Acad. Sci. URSS (N.S.) **18**, 137–140 (1938)] and (2) is an analogue of a theorem of Rogosinski [Compositio Math. **3**, 199–226 (1936)].

S. E. Warschawski (St. Louis, Mo.).

Khajalia, G. Sur la théorie de la représentation conforme des domaines doublement connexes. Rec. Math. [Mat. Sbornik] N.S. **8** (50), 97–106 (1940). (Russian. French summary) [MF 3192]

Let D be a doubly-connected, finite plane domain bounded internally and externally by the boundary continua C_2 and C_1, respectively. Let $\{f(z)=u+iv\}$ be the family of functions which are regular in D and satisfy the condition that

$$(1)\qquad \lim_{C\to C_2}\frac{1}{2}\int_C\{udv-vdu\}=\pi,$$

where C is any smooth closed curve of D not homologous to zero. Under these conditions the minimum of the integral

$$(2)\qquad \iint_D|f'(z)|^2d\sigma,$$

where $d\sigma$ is an element of area of D, will be attained if and only if f maps D on a ring $\Re$: $1<|w-w_0|<R$. If D is accessible from without, then a sequence of minimal rational fractions converges uniformly in D to the function f which maps D on $\Re$. The calculation of the coefficients of these fractions is reducible to the solution of a system of linear homogeneous equations. Problems of a similar type have been considered by Zarankiewicz [Z. Angew. Math. Mech. **14**, 87–104 (1934)]; and by Kufareff [Bull. Inst. Math. Mec. Univ. Koubycheff Tomsk **1**, 228–236 (1937)]. Kufareff, for example, finds the minimum of (2) subject to the condition that $|f'(t)|=1$, where t is an interior point of D, the calculation of the extremal in this case being effected with the aid of elliptic functions.

D. C. Spencer.

Courant, R., Manel, B. and Shiffman, M. A general theorem on conformal mapping of multiply connected domains. Proc. Nat. Acad. Sci. U. S. A. **26**, 503–507 (1940). [MF 2638]

In the theory of conformal mapping of domains of connectivity k, a basic problem is the determination of types of domains B which depend on a finite number of parameters such that any domain of connectivity k can be mapped conformally on one of them. Such a class of domains may be called a class of normal domains. Methods previously developed for the solution of the problems of Plateau and Douglas are now used to show that a wide variety in the choice of normal domains is possible; it is shown, namely, that the domains consisting of the exterior of k non-intersecting curves which are respectively similar under expansions and translations (that is, under transformations of the form $x'=ax+b$, $y'=ay+c$) to any k given convex analytic curves form a class of normal domains.

E. F. Beckenbach (Ann Arbor, Mich.).

Garwick, Jan V. Über das Typenproblem. Arch. Math. Naturvid. **43**, 33–46 (1940). [MF 1396]

This paper contains a contribution to Speiser's problem of finding connections between the type of a simply connected Riemann surface over the z-plane and the distribution of its branch points. Various necessary or sufficient conditions for the parabolic type have been given [see R. Nevanlinna, Eindeutige analytische Funktionen, XII, Berlin, 1936]. The author proves here a sufficient condition, which is a special case of a theorem of Wittich (not yet published), by restricting himself to surfaces with only logarithmic branch points over a finite number q of points of the z-plane. According to a procedure of Kakutani [see Jap. J. Math. **13**, 393–404 (1937)] he constructs a quasi-conformal mapping of the given surface using Speiser's topological tree as skeleton of the construction, and by studying the transformation finds the theorem: Let $\sigma(n)$ be the number of branches of the topological tree in the first n generations; the divergence of $\sum 1/\sigma(n)$ is a sufficient condition for the parabolic type of the given surface. For $q=3$ the theorem coincides with a theorem of R. Nevanlinna [Comment. Math. Helv. **5**, 95–107 (1933)].

C. Loewner (Louisville, Ky.).

Nevanlinna, Rolf. Ein Satz über offene Riemannsche Flächen. Ann. Acad. Sci. Fennicae (A) **54**, no. 3, 16 pp. (1940). [MF 2210]

According to the fundamental theorem of conformal mapping, an open simply-connected Riemann surface can be mapped conformally either on the schlicht punctured plane (parabolic type) or on the schlicht interior of a finite circle. The "problem of type" is the problem of distinguishing between these two cases; it is usually attacked by considering a surface as covering surface of the complex plane, the type being determined in its dependence on the branchedness of the surface relative to the plane. A related problem is the question of the vanishing of the capacity of a set of points embedded in the plane. Both problems are special cases of the problem of determining when an open Riemann surface of arbitrary connectivity has a "null-boundary." Let the open Riemann surface F be exhausted with a sequence of nested domains F_n $(n=0, 1, 2, \cdots)$, which are each bounded by a finite number of analytic arcs Γ_n. For each $n=1, 2, \cdots$, consider the harmonic mass function $\omega(\Gamma_n, P, F_n-F_0)$ of the boundary Γ_n, measured at the point P of F_n-F_0; on the subsurface F_n-F_0, ω is harmonic and $\omega=1$ on Γ_n, $\omega=0$ on Γ_0. With increasing n, ω decreases monotonically and there exists the limiting function $\omega(\Gamma, P, F-F_0)$, the harmonic mass of the ideal boundary Γ of F. Either $\omega\equiv 0$ or ω is positive at each interior point of $F-F_0$, this division into cases being independent of the choice of the approximating sequence $\{F_n\}$. If $\omega\equiv 0$, we say that F has a null-boundary Γ, otherwise a positive boundary.

The author obtains quite directly from Bieberbach's area theorem a simple but powerful criterion (sufficient condition) for the vanishing of the boundary of an open Riemann surface defined directly in terms of conformally connected surface elements. This criterion is extensively applicable to each of the two first mentioned problems. Let the plane be covered with a net Q_0 of squares with length of side 1, let these squares be subdivided into a net Q_1 of squares with length of side 1/2, etc. Let E be a closed and bounded set of the plane and let m_n be the number of squares of Q_n having an interior or boundary point in common with E. The author shows that, if the series $\sum_{n=0}^{\infty}(1/m_n)$ diverges, then the capacity of E vanishes; for it follows from his criterion that the open complementary set of E has a null-boundary. The criterion allows the consolidation and generalization of known sufficient conditions for the parabolic type of Riemann surface, whereby for surfaces which are not simply-connected the notion of parabolic type is naturally replaced by the notion of null-boundary. In particular, sufficient conditions of Ahlfors and the more recent quite general sufficient condition of Wittich [Math. Z. **45**, 642–668 (1939); these Rev. **1**, 211] may be established and generalized as consequences of the present result.

E. F. Beckenbach (Ann Arbor, Mich.).

Stoïlow, S. Sur les surfaces de Riemann normalement exhaustibles et sur le théorème des disques pour ces surfaces. Compositio Math. **7**, 428–435 (1940). [MF 1873]

In distinction from the metrico-topological notion of regular exhaustion, which is fundamental in the general covering theory of Ahlfors, and which leads in particular to properties concerning the distribution of values of meromorphic functions (and of any other functions whose Riemann surfaces are regularly exhaustible), the author discusses a purely topological normal exhaustion and obtains results concerning the distribution of values for functions whose Riemann surfaces are normally exhaustible. An open Riemann surface (R), defined by the two-dimensional variety V and the function $f(V)$ giving an interior map of (R) on a Euclidean sphere, is normally exhaustible if there exists on V a sequence of polyhedric domains D_i such that $D_i\subset D_{i+1}$ in the strict sense, $\sum_i D_i=V$, and such that the transformation of D_i into $f(D_i)$ is an interior transformation. The class of normally exhaustible surfaces is not contained in, nor does it contain, the class of regularly exhaustible surfaces. An entire function of order less than $\frac{1}{2}$ has a normally exhaustible surface of parabolic type, while the Lusin-Privaloff function has a normally exhaustible surface of hyperbolic type. For a simply connected normally exhaustible Riemann surface, the lacunary set, that is, the set of points of the sphere which are not covered by any sheet of the surface, reduces to a single point (surface of the first kind) or constitutes a continuum (surface of the second kind). Since regularly exhaustible surfaces have at most two lacunary points, normally exhaustible surfaces of the second kind are not regularly exhaustible; but those of the first kind are regularly exhaustible.

The author obtains the following result analogous to Ahlfors' disc theorem concerning entire functions; but while three is the least number of discs that will serve in Ahlfors' theorem, two suffice here: If (R) is a simply connected normally exhaustible Riemann surface, and if on the sphere on which (R) is represented D_1 and D_2 are two non-overlapping Jordan domains (discs) having no point in common with the lacunary set of (R), then one at least of these domains is covered by a simple sheet of (R), and indeed by an infinite number of such sheets unless (R) consists of only a finite number of sheets. *E. F. Beckenbach.*

Behnke, H. und Stein, K. Die Sätze von Weierstrass und Mittag-Leffler auf Riemann'schen Flächen. Vierteljschr. Naturforsch. Ges. Zürich **85**, 178–190 (1940). [MF 2705]

Verfasser übertragen die Sätze von Weierstrass und Mittag-Leffler über die Existenz einer in der endlichen komplexen Ebene analytischen bzw. meromorphen Funktion zu dort vorgegebenen Nullstellen bzw. Polstellen mit ihren Hauptteilen, auf alle endlichen, endlich-blättrigen, im Innern unverzweigten Riemannsche Flächen. Und zwar wird diese Übertragung als Sonderfall entsprechender Sätze aus der Funktionentheorie mehrerer komplexer Veränderlichen gewonnen. Beim Beweise stützen sich Verfasser auf einen Satz von K. Oka (bzw. von K. Oka und K. Stein) über die Existenz einer in einem beliebigen endlichen, schlichten Regularitätsbereiche $\mathfrak{B}$ des Raumes R_{2n} meromorphen Funktion $f(z_1, \cdots, z_n)$ zu in $\mathfrak{B}$ vorgegebenen Polflächen, die in $\mathfrak{B}$ den bekannten Verträglichkeitsbedingungen genügen, bzw. über die Existenz einer in $\mathfrak{B}$ regulären Funktion zu in $\mathfrak{B}$ vorgegebenen Nullstellenflächen, die dort außer den Verträglichkeitsbedingungen noch einer gewissen topologischen Bedingung genügen [vgl. auch die Literatur über die Cousin'schen Sätze]. Die genannten Sätze werden zunächst auf nicht-schlichte, endlichblättrige, im Innern nicht verzweigte Regularitätsbereiche des R_{2n} übertragen, und schließlich wird nachgewiesen, daß für $n=1$ jede endliche, endlichblättrige, im Innern nicht verzweigte Riemannsche Fläche ein Regularitätsbereich (das heisst, Existenzbereich einer analytischen Funktion) ist. *P. Thullen* (Quito).

Behnke, H. und Stein, K. Die Konvexität in der Funktionentheorie mehrerer komplexer Veränderlichen. Mitt. Math. Ges. Hamburg **8**, 34–81 (1940). [MF 2706]

Die Arbeit gibt einen systematischen Überblick über die

verschiedenen Begriffe der Konvexität inbezug auf analytische Funktionen und Flächen, die in der neueren Funktionentheorie von n komplexen Veränderlichen, insbesondere bei der Untersuchung der Regularitäts- und Konvergenzbereiche eine fundamentale Rolle spielen. Verfasser gehen dabei von der elementar-geometrischen Konvexität in der Ebene und in höherdimensionalen Räumen aus, deren Darstellung so gefaßt ist, daß sie als Vorlage für die später entwickelten Konvexitäten dient. Im Raume R_{2n} von n komplexen Veränderlichen wird zunächst die "Planarkonvexität" behandelt, die im wesentlichen die gleichen Eigenschaften aufweist wie die elementar-geometrische Konvexität; insbesondere zieht auch hier die Konvexität "im kleinen" notwendig die "im großen" nach sich. Nach dieser Einführung gehen Verfasser zu der Betrachtung der Konvexität inbezug auf Monome: $Cw^\alpha z^\beta$ über, welche die maximalen Konvergenzbereiche von Potenz- und Laurentreihen von zwei komplexen Veränderlichen w, z charakterisiert. Es ist das der einfachste, nicht lineare Fall der Regulärkonvexität, die allgemein so definiert wird: Ein Bereich $\mathfrak{B}$ über dem Raume R_{2n} heißt regulärkonvex, falls es zu jedem ganz in $\mathfrak{B}$ gelegenen Bereich $\mathfrak{B}_0$ einen Bereich $\mathfrak{B}_0^* \subset\subset \mathfrak{B}$ gibt und zu jedem Punkte P in $\mathfrak{B}$, außerhalb $\mathfrak{B}_0^*$, eine in $\mathfrak{B}$ reguläre Funktion $f(z_1, \cdots, z_n)$ mit $|f(P)| > \max|f(\mathfrak{B}_0)|$. Die Regulärkonvexität ist bekanntlich die charakteristische Eigenschaft der Regularitätsbereiche (Existenzbereiche regulärer Funktionen). Zu jedem Bereiche $\mathfrak{B}$ gibt es eine Regularitätshülle $\mathfrak{H}(\mathfrak{B})$, als Durchschnitt aller $\mathfrak{B}$ umfassenden Regularitätsbereiche und zugleich als der kleinste $\mathfrak{B}$ umfassende Regularitätsbereich. Bei der Besprechung der Eigenschaften der Regularitätshüllen wird insbesondere auf die Aproximierbarkeit der Regularitätsbereiche und den dabei auftretenden Begriff der "Nebenhülle" eingegangen. Der Spezialfall der Hartogs'schen Körper wird gesondert behandelt als der der Konvergenzbereiche von Entwicklungen regulärer Funktionen nach nur einer Veränderlichen: $\sum_{m=0}^{\infty}(z_1-z_1^{(0)})^m f_m(z_2, \cdots, z_n)$. Ein Hartogs'scher Körper mit der Achse $z_1=z_1^{(0)}$ ist dann und nur dann ein solcher Konvergenzbereich (oder Regularitätsbereich), falls er konvex inbezug auf Funktionen $(z_1-z_1^{(0)})^m f(z_2, \cdots, z_n)$ ist. Wichtig ist, daß für $n=2$ ein schlichter Hartogs'scher Regularitätsbereich konvex inbezug auf rationale Funktionen ist, bzw. inbezug auf Polynome, falls er zudem einfach-zusammenhängend ist. Auf Grund eines Satzes von K. Oka und unter Zuhilfenahme des Begriffes der "regulär vollständigen $\mathfrak{J}$-Familie" bzw. der "meromorph vollständigen $\mathfrak{R}$-Familie" läßt sich ein enger Zusammenhang herstellen zwischen der Regulärkonvexität und der Frage nach der Entwickelbarkeit aller in einem Bereiche $\mathfrak{B}$ regulären Funktionen nach Funktionen, die in einem $\mathfrak{B}$ umfassenden Bereiche regulär sind. Es gilt: Dann und nur dann lassen sich alle in einem Regularitätsbereiche $\mathfrak{B}$ regulären Funktionen nach Funktionen einer $\mathfrak{J}$-Familie (bzw. $\mathfrak{R}$-Familie) entwickeln, falls $\mathfrak{B}$ konvex inbezug auf die gegebene Familie ist. Zu jedem Bereiche $\mathfrak{B}$ und einer in ihm regulären $\mathfrak{J}$-Familie gibt es eine $\mathfrak{J}$-konvexe Hülle und eine $\mathfrak{J}$-konvexe Nebenhülle, die gleiche oder ähnliche Eigenschaften wie die Regularitätshüllen und Regularitätsnebenhüllen besitzen. Der einfachste und wichtigste Fall der $\mathfrak{J}$-Konvexität bzw. $\mathfrak{R}$-Konvexität ist die Konvexität inbezug auf Polynome (Rungesche Bereiche) bzw. die inbezug auf rationale Funktionen. Eine geometrische Formulierung der $\mathfrak{R}$-Konvexität ergibt sich aus der Tatsache, daß ein Bereich $\mathfrak{B}$ konvex inbezug auf eine $\mathfrak{R}$-Familie: $\mathfrak{R}(f)$ ist, falls es zu jedem ganz in $\mathfrak{B}$ liegenden Bereich $\mathfrak{B}_0$ einen Bereich $\mathfrak{B}_0^* \subset\subset \mathfrak{B}$ gibt und durch jeden Punkt P in $\mathfrak{B}$, außerhalb $\mathfrak{B}_0^*$, eine analytische Fläche: $g(z_1, \cdots, z_n)=0$ läuft, die nicht in $\mathfrak{B}_0$ eindringt; g aus der Familie $\mathfrak{R}(f)$. Bisher wurden wegen der möglichen Nicht-Schlichtheit der vorkommenden Bereiche deren Rand außer Betracht gelassen. Falls nun ein Bereich sich in einen ihn ganz umfassenden Bereich einbetten läßt, so fragt es sich, wie die Regulärkonvexität sich durch Eigenschaften des Randes ausdrücken läßt. Man wird hierbei auf die Begriffe der Pseudokonvexität und der lokal-analytischen Konvexität der Randhyperfläche inbezug auf analytische Flächen geführt. Doch ist das gestellte Problem, vor allem die Frage, ob die Konvexität des Randes im kleinen für die im großen hinreichend ist, nur in Einzelfällen gelöst. Es sei bemerkt, daß der vorliegende Bericht auch ohne tiefere Kenntnis der Funktionentheorie mehrerer komplexen Veränderlichen verstanden werden kann.

P. Thullen (Quito).

Kasner, Edward. Conformality in connection with functions of two complex variables. Trans. Amer. Math. Soc. **48**, 50–62 (1940). [MF 2504]

Verfasser untersucht charakteristische Eigenschaften geometrischer Art der analytischen Abbildungen im Raume R_4 zweier komplexen Veränderlichen. Der Hauptsatz besagt, dass eine Abbildung im R_4 dann und nur dann analytisch ist, falls sie jeden "Pseudowinkel," der von irgendeinem Kurvenstück C und einem beliebigen es in einem Punkte P schneidenden Hyperflächenstück H gebildet wird, invariant lässt. Der Pseudowinkel zwischen C und H im gemeinsamen Schnittpunkte P ist dabei der von C und einer ebenfalls durch P laufenden Hilfskurve C' gebildete Winkel, wobei C' dadurch bestimmt ist, dass sie sowohl H wie auch die (eindeutig bestimmte) analytische Ebene berührt, die durch P geht und C dort berührt. Ferner beweist Verfasser, dass eine analytischer Transformation, welche vier zweifach unendliche Mannigfaltigkeiten analytischer Ebenen wieder in solche Ebenen überführt, notwendig eine projektive Transformation des R_4 in sich ist. (Bei der Beurteilung der Arbeit muss berücksichtigt werden, dass ihre Entstehung um mehr als 30 Jahre zurückliegt.) *P. Thullen* (Quito).

Martin, W. T. On a minimum problem in the theory of analytic functions of several variables. Trans. Amer. Math. Soc. **48**, 351–357 (1940). [MF 2742]

In 1932 Wirtinger posed and solved the problem of finding an analytic function $f(z)$ that best approximates, in the mean-square sense, a complex-valued function $\phi(z, \bar{z}) \equiv \phi(x+iy, x-iy)$, in a given region $\mathfrak{G}$ in the z-plane, where ϕ is continuous and has continuous first partial derivatives in $\mathfrak{G}$ with respect to x and y [Monatsh. Math. Phys. **29**, 377–384 (1932)]. The solution is obtained by employing Green's function. Later [Monatsh. Math. Phys. **4**, 426–431 (1939)] he solved the analogous problem, using multiple Fourier series, for n complex variables for a $2n$-dimensional hypersphere H and for the function $\phi(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)$ merely integrable over H. In both cases he obtains unique solutions. He conjectured that the problem has a solution for more general regions and for more general functions, but that the solution probably would involve extensions of Green's functions. Now in several complex variables Bergman has been able to replace Green's functions by complex orthogonal functions and a kernel of a region [Math. Z. **29**, 640–677 (1929); J. Reine Angew. Math. **169**, 1–42 (1933)].

By using a complete system of orthogonal functions and the kernel for a region, introduced by Bergman [loc. cit.],

the author solves the problem for regions B, for which there exists a complete set of orthogonal functions belonging to L^2 (for all simply-connected bounded regions) and for $\phi(z_1, z_2, \bar{z}_1, \bar{z}_2) \epsilon L^2(B)$. The solution is a unique minimum over the class of analytic functions of integrable square. He determines f explicitly for several special regions. The author also indicates that analogous problems may be solved: first, when the region of integration is a two-dimensional surface on the four-dimensional boundary of B, when B has a "distinguished boundary surface"; and second, when f is not only analytic and integrable square over B but also takes on specific values at specific points of B. The case of n complex variables involves no essential changes.

A. Gelbart (Raleigh, N. C.).

Spampinato, Nicolò. Intorno alle funzioni monogene di una classe speciale di variabile ipercomplessa introdotte dal Valerias. Boll. Un. Mat. Ital. (2) **2**, 348–349 (1940). [MF 2988]

Anghelutza, Th. Une remarque sur une généralisation des fonctions analytiques. Mathematica, Cluj **16**, 53–56 (1940). [MF 2483]

The purpose of this note is to show that the generalization of analytic functions considered by Nicolesco [Thèse, p. 86, Gauthier-Villars, Paris, 1928; Mathematica, Cluj **3**, 134–142 (1930); **14**, 18–20 (1938)] can be obtained from the generalization given by Beltrami by means of a conformal transformation [Ann. Mat. Pura Appl. (2) **1**, 329–366 (1867)]. *W. T. Martin* (Cambridge, Mass.).

Rosenhead, L. Note on the expansion of the Jacobian elliptic functions in powers of k^2 and $1-k^2$. J. London Math. Soc. **15**, 7–10 (1940). [MF 2577]

The Jacobian elliptic function sn u is, for purposes of numerical computation, expanded in powers of k^2 and $1-k^2$. Writing $\mathrm{sn}^{-1}(\sin\phi)$ as an ordinary elliptic integral and expanding the latter one gets $\sin\phi = \mathrm{sn}(\phi+\lambda)$, where λ is determined by a transcendental equation, which can be solved by iterations. *W. Feller* (Providence, R. I.).

Neville, E. H. Expansion of Jacobian functions in powers of the moduli. J. London Math. Soc. **15**, 113–115 (1940). [MF 2520]

A simple derivation of formulas for the derivatives of the Jacobian elliptic functions with respect to the square of the modulus is given. By iteration the derivatives of higher order may be found. These involve the functions

$$I_{2m}(u) = \int_0^u \left(\frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}\right)^{2m} du,$$

and a recurrence relation is given for the determination of these functions. According to the author the formula with which he starts is a rediscovery of a result of Hermite which has been overlooked by later writers.

H. S. Zuckerman (Seattle, Wash.).

Petersson, Hans. Über eine Metrisierung der automorphen Formen und die Theorie der Poincaréschen Reihen. Math. Ann. **117**, 453–537 (1940). [MF 3095]

Let Γ be a group of linear substitutions

$$\tau \rightarrow \frac{\alpha\tau+\beta}{\gamma\tau+\delta} = M\tau$$

with real coefficients $\alpha, \beta, \gamma, \delta$ and $\alpha\delta-\beta\gamma=1$, which has a finite number of generators and contains parabolic substitutions. Let F be a fundamental domain of Γ in the upper half-plane $\mathfrak{J}(\tau) \geqq 0$. A function $\phi(\tau)$ is called an automorphic form of the class $\{\Gamma, -r, v\}$ if it has the following two properties: (1) in any point of F, the function $\phi(\tau)$ is meromorphic with respect to the uniformizing local variable; (2) for any substitution M of Γ, the functional equation $\phi(M\tau) = v(M)(\gamma\tau+\delta)^r\phi(\tau)$ holds, where r is a constant and the multiplicator $v(M)$ depends only upon M. Suppose moreover that $r>2$ and $|v(M)|=1$. Generalizing the Eisenstein series from the theory of elliptic modular functions, Poincaré found a method of representing certain automorphic forms by an infinite series of elementary functions. Later Poincaré, Ritter and Fricke proved that any automorphic form can be expressed as a linear combination of Poincaré series; but their proof was not constructive.

The author obtains a new proof of this theorem, which is more satisfactory also from the practical point of view. He defines the scalar product (ϕ, ψ) of two automorphic forms $\phi(\tau)$ and $\psi(\tau)$ of the same class by the formula

$$(\phi, \psi) = \iint_F \phi(\tau)\overline{\psi(\tau)}y^r \frac{dxdy}{y^2}, \qquad \tau = x+iy.$$

Using the invariance of the hyperbolic measure in the group Γ, he calculates this scalar product if ϕ is an integral automorphic form vanishing in all the parabolic vertices of F and ψ a Poincaré series. The result gives a new independent definition of the Poincaré series, by their inner properties, and an explicit basis for all automorphic forms of the class $\{\Gamma, -r, v\}$. In particular, if Γ is a congruence-subgroup of the modular group, the theory still holds in the limiting case $r=2$, $v(M)=1$; but the proofs are more difficult. In this case, $\phi(\tau)d\tau$ becomes an Abelian differential on the Riemann surface F and (ϕ, ψ) is a bilinear form of the periods of Abelian integrals, closely connected to a well-known relationship from Riemann's work on Abelian functions.

Since the results contain a considerable simplification of the former theory of automorphic functions, they will be important for further research. *C. L. Siegel.*

Maass, Hans. Zur Theorie der automorphen Funktionen von n Veränderlichen. Math. Ann. **117**, 538–578 (1940). [MF 3096]

Let Γ be a group of "hyperabelian" transformations

$$\tau^{(k)} \rightarrow \frac{\alpha^{(k)}\tau^{(k)}+\beta^{(k)}}{\gamma^{(k)}\tau^{(k)}+\delta^{(k)}}$$

with real coefficients $\alpha^{(k)}, \beta^{(k)}, \gamma^{(k)}, \delta^{(k)}$ and $\alpha^{(k)}\delta^{(k)}-\beta^{(k)}\gamma^{(k)}=1$ $(k=1, \cdots, n)$. In his paper "Über Gruppen von hyperabelschen Transformationen" [S.-B. Heidelberger Akad. Wiss. Math.-Naturwiss. Kl. **1940**, 2. Abhandlung], the author considered the problem of constructing a fundamental domain F for Γ in the $2n$-dimensional space $\mathfrak{J}(\tau^{(k)}) \geqq 0$ and applied his results to the special case of Hilbert's modular group Γ_0, that is, the group of the unimodular substitutions

$$\tau \rightarrow \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$$

in a totally real algebraic field of degree n. In the present paper he begins the study of the automorphic functions with the group Γ. A function $\phi(\tau)$ of the variables $\tau^{(1)}, \cdots, \tau^{(n)}$ is called an integral automorphic form of the class $\{\Gamma, -r, v\}$

if $\phi(\tau)$ is regular in all points of the fundamental domain F and satisfies the relationship

$$\phi(M\tau)=v(M)\phi(\tau)\prod_{k=1}^{n}(\gamma^{(k)}\tau^{(k)}+\delta^{(k)})^{r}$$

for all transformations M of Γ, where r is a constant and the multiplicator $v(M)$ depends only upon M. If Γ is restricted by several conditions, which are fulfilled in the case $\Gamma=\Gamma_n$, and if $r>2$, $|v(M)|=1$, special automorphic forms of the class $\{\Gamma, -r, -v\}$ can be constructed by a generalization of Poincaré series. The important theorem is proved, that any integral automorphic form of this class is a linear combination of those Poincaré series. The proof uses a generalization of the scalar product of two automorphic forms introduced by H. Petersson. The automorphic functions are defined as the quotients of automorphic forms of the same class. The author obtains the result that any automorphic function is a rational function of $n+1$ fixed automorphic functions, which are connected by a single algebraic equation. *C. L. Siegel* (Princeton, N. J.).

Selberg, Atle. Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist. Arch. Math. Naturvid. **43**, 47–50 (1940). [MF 3238]

This paper considers two modular forms

$$f(\tau)=\sum_{n=1}^{\infty}\alpha_n e^{2\pi i n\tau},\quad \phi(\tau)=\sum_{n=1}^{\infty}\beta_n e^{2\pi i n\tau},$$

of positive dimension k, and the Dirichlet series

$$\zeta(2s)\sum_{n=1}^{\infty}\frac{\alpha_n\bar{\beta}_n}{n^{k-1+s}}$$

generated by their coefficients. The Dirichlet series can be expressed in terms of a double integral involving the Epstein zeta-function and it can then be defined over the whole s-plane. Some inequalities concerning the coefficients of the modular forms are given. These include improvements of inequalities of Salié and Walfisz. Only a special case is discussed and the proofs briefly described. The author states that his general results with complete proofs will appear later. *H. S. Zuckerman* (Seattle, Wash.).

Theory of Series

Auluck, F. C. On some theorems of Ramanujan. Proc. Indian Acad. Sci., Sect. A. **11**, 376–378 (1940). [MF 2748]

The author proves certain properties of the function

$$\phi(x)=\sum_{n=1}^{\infty}\frac{n^{n-2}x^{n-1}}{(n-1)!}e^{-nx}$$

which were stated in Question 738 of Ramanujan's Collected Papers [Cambridge, 1927]. Additional properties of the derivative $\phi'(x)$ are obtained which enable the author to prove the statement in Question 526 concerning the series $\sum_{m=1}^{\infty}m^{m-2}/(n+m)^m$. *P. W. Ketchum* (Urbana, Ill.).

Chowla, S. and Auluck, F. C. An approximation connected with exp. x. Math. Student **8**, 75–77 (1940). [MF 3283]

The authors prove that

$$\lim e^{-x}\sum_{k=0}^{[x]}\frac{x^k}{k!}=\frac{1}{2}.$$

P. Erdös (Philadelphia, Pa.).

Störmer, Carl. Sur une généralisation de la constante d'Euler. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 316–319. [MF 3524]

The author defines a function $D(T)$ such that

$$D(T)=\lim_{n\to\infty}\left[\frac{\pi}{T}\operatorname{ctg}\frac{\pi}{T}+\frac{\pi}{T}\operatorname{ctg}\frac{2\pi}{T}+\cdots+\frac{\pi}{T}\operatorname{ctg}\frac{n\pi}{T}-\ln\left(\frac{T}{\pi}\sin\frac{n\pi}{T}\right)\right].$$

He develops an asymptotic formula for $D(T)$ in terms of Euler's constant γ and the Bernoulli numbers. By means of this formula he proves

$$\lim_{|T|\to\infty}D(T)=\gamma.$$

T. Fort (Bethlehem, Pa.).

Garabedian, H. L. A new formula for the Bernoulli numbers. Bull. Amer. Math. Soc. **46**, 531–533 (1940). [MF 2425]

Let $B_1, B_2, \cdots$ denote Bernoulli numbers and let $a_n=(1+n)^k$, $k=0, 1, 2, \cdots$. By use of the Cesàro and Abel methods of summability of the divergent series $1-2^k+3^k-4^k+\cdots$, it is shown that for each $k=0, 1, 2, \cdots$

$$B_{k+1}=\frac{(-1)^{k+1}(k+1)}{2^{k+1}-1}\sum_{i=0}^{k}\frac{\Delta^i a_0}{2^{i+1}}.$$

R. P. Agnew (Ithaca, N. Y.).

Guareschi, Giacinto. Espressione dei numeri di Bernoulli mediante funzioni simmetriche complete. Boll. Mat. (4) **1**, 17–19 (1940). [MF 1799]

The author states that the Bernoullian numbers B_l can be represented in the form

$$(1)\quad B_l=l!\sum_{\beta_1+2\beta_2+\cdots+l\beta_l=l}\frac{(-1)^{l+\beta_1+\cdots+\beta_l}(\beta_1+\cdots+\beta_l)!}{(\beta_1!\cdots\beta_l!)((2!)^{\beta_1}(3!)^{\beta_2}\cdots((l+1)!)^{\beta_l})}=c_l(x_1, \cdots, x_l),$$

where $c_l(x_1, \cdots, x_l)$ denotes the "complete symmetrical function" of $x_1, \cdots, x_l$, that is, the sum of all power products of $x_1, \cdots, x_l$ of degree l (also: Wronski's alef-function), and where $x_1, \cdots, x_l$ are the roots of

$$x^l-\frac{1}{2!}x^{l-1}+\frac{1}{3!}x^{l-2}-\cdots+(-1)^l\frac{1}{(l+1)!}=0.$$

This is an immediate consequence of the following remark: if

$$\sum_0^{\infty}q_lx^l\sum_0^{\infty}p_lx^l=1,\qquad q_0=p_0=1,$$

then

$$q_l+q_{l-1}p_1+\cdots+q_1p_{l-1}+p_l=0,\quad l=1, 2, \cdots,$$

and from this it follows [see, for example, Enzyklopädie der mathematischen Wissenschaften I_1, 464–465] that $q_l=c_l(x_1, \cdots, x_l)$, where $x_1, \cdots, x_l$ are the roots of $x^l+p_1x^{l-1}+\cdots+p_l=0$, and

$$q_l=\sum_{\beta_1+2\beta_2+\cdots+l\beta_l=l}(-1)^{\beta_1+\cdots+\beta_l}\frac{(\beta_1+\cdots+\beta_l)!}{\beta_1!\cdots\beta_l!}p_1^{\beta_1}\cdots p_l^{\beta_l}.$$

Applying this to

$$\sum_{l=0}^{\infty}\frac{B_l}{l!}x^l\cdot\frac{1-e^{-x}}{x}=\sum_{l=0}^{\infty}\frac{B_l}{l!}x^l\sum_{l=0}^{\infty}(-1)^l\frac{x^l}{(l+1)!}=1,$$

one obtains (1). *F. A. Behrend* (London).

Obrechkoff, Nikola. Sur les moyennes arithmétiques de la série de Taylor. C. R. Acad. Sci. Paris **210**, 526–528 (1940). [MF 2262]

The author derives a formula for the arithmetic means of the partial sums of a Taylor series; applications are announced. *O. Szász* (Cincinnati, Ohio).

Daniell, P. J. Ratio tests for double power series. Quart. J. Math., Oxford Ser. **11**, 183–192 (1940). [MF 3128]

Let $\sum c_{s,t}x^s y^t$ be a double power series in which x, y, and the coefficients $c_{s,t}$ are positive. Let, when $0 \leqq p \leqq 1$,

$$f(p) = \lim_{m\to\infty;\, \epsilon\to 0} \ \text{least upper bound}_{s+t\geqq m;\, |s/(s+t)-p|\leqq\epsilon} (s+t)^{-1} \log c_{st}.$$

The series is convergent at (x, y) if

$$p \log x + (1-p) \log y + f(p) < 0, \qquad 0 \leqq p \leqq 1,$$

and is divergent at (x, y) if

$$p \log x + (1-p) \log y + f(p) > 0$$

for at least one p in the interval $0 \leqq p \leqq 1$. In case certain limits are uniform, simpler formulas suffice to give $f(p)$. *R. P. Agnew* (Ithaca, N. Y.).

Mall, Josef. Beitrag zur Theorie der mehrdimensionalen Padéschen Tafel. Math. Z. **46**, 337–349 (1940). [MF 2403]

Given a power series $f(x) = \sum_0^\infty c_n x^n$, $c_0 \neq 0$, and a pair of integers $\mu \geqq 0$, $\nu \geqq 0$. Then there exists a unique pair of relatively prime polynomials $P_{\mu\nu}(x)$, $Q_{\mu\nu}(x)$ such that $P_{\mu\nu}(0) = c_0$, $Q_{\mu\nu}(0) = 1$ and for a certain $\lambda \geqq 0$, $x^\lambda P$ is of degree ν, $x^\lambda Q$ of degree not greater than μ and such that

$$x^\lambda \{Q_{\mu\nu}(x) \sum_0^\infty c_n x^n - P_{\mu\nu}(x)\}$$

starts with the term $ax^{\mu+\nu+1}$. This system of polynomials is known as the Padé table. Under certain assumptions $P_{\mu\nu}(x)/Q_{\mu\nu}(x) \to f(x)$. [Cf. O. Perron, Die Lehre von den Kettenbrüchen, 2nd ed., chap. 10; H. S. Wall, Bull. Amer. Math. Soc. **38**, 752–760 (1930).] In his thesis [Munich, 1934] the author generalized this concept to $n+1$ dimensions for n simultaneously given power series. In the present paper a convergence criterion is given improving on one of his previous results. *O. Szász* (Cincinnati, Ohio).

Agnew, Ralph Palmer. On rearrangements of series. Bull. Amer. Math. Soc. **46**, 797–799 (1940). [MF 2925]

Let E be the space of all permutations of positive integers and let the distance between two permutations $(x_1, x_2, \cdots)$ and $(y_1, y_2, \cdots)$ be defined by the formula

$$(x, y) = \sum_{n=1}^\infty \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

Let furthermore $\sum_{n=1}^\infty c(n)$ be an arbitrary convergent series for which $\sum_{n=1}^\infty |c(n)| = \infty$. It is proved that for each permutation x, except those belonging to a set of the first category, one has

$$\liminf_{N\to\infty} \sum_{n=1}^N c(x_n) = -\infty, \quad \limsup_{N\to\infty} \sum_{n=1}^N c(x_n) = \infty.$$

In order to make the result meaningful it is also proved that E is of the second category. This last statement requires a separate proof in view of the fact that E is not a complete space. *M. Kac* (Ithaca, N. Y.).

Lyra, Gerhard. Über den Zusammenhang einiger Reihensätze. Math. Z. **46**, 627–634 (1940). [MF 2786]

This paper is concerned with necessary and sufficient conditions for the Cesàro summability of series. The author points out a connection between a theorem of Knopp and a theorem of Hardy-Littlewood in both of which necessary and sufficient conditions for Cesàro summability are determined in terms of other limit processes. He then proceeds to generalize both theorems and to point out a connection between his generalized theorems similar to the connection between the quoted theorems of Knopp and Hardy-Littlewood. *T. Fort* (Bethlehem, Pa.).

Lyra, Gerhard. Über einen Satz zur Theorie der C-summierbaren Reihen. Math. Z. **45**, 559–572 (1939). [MF 409]

In this paper the author obtains a theorem in Cesàro summability closely related to theorems of Hardy and Littlewood, and Knopf, all of which were proved in 1924. The author introduces the notion of remainders of a C_k summable series $\sum a_n$. A remainder $R_n^{(r)}$ of the rth order is the C_k sum of $\sum_{r=n}^\infty R_r^{(r-1)}$ for $r \geqq 1$, and $R_n^{(r)} = a_n$ for $r = 0$. The theorem then reads: Let $r \geqq 0$, $p \geqq 0$, and $r \geqq p+1$. Then a necessary and sufficient condition for the C_p summability of a series $\sum a_n$ to s is that

$$\sum_{n=0}^\infty a_n / \binom{n+r}{r}$$

is C_{p-r} summable, that the sequence of the rth order remainders $R_n^{(r)}$ of the C_{p-r} summable series exists, and that $\sum_{n=0}^\infty R_n^{(r)}$ is C_{p-r} summable to s. *N. Levinson*.

Hyslop, J. M. Note on certain related conditions in the theory of Cesàro summability. Proc. Edinburgh Math. Soc. (2) **6**, 166–171 (1940). [MF 2850]

This paper is a continuation of an earlier one in the same journal [(2) **5**, 182–201 (1938)]. Let $0 < \rho < 1$, let p be a positive integer and let $S_n^{(p)}/A_n^{(p)}$ denote the (C, p) transform of a series $\sum a_n$. If $\sum n^\rho a_n$ is summable (C, p) to s, then $S_n^{(p)} = s' A_n^{(p)} + \beta_n' n^{p-\rho}$, where $\beta_n' \to 0$, $\sum n^{-1}\beta_n'$ is convergent and s' a constant for which a formula is given. Conversely, if $S_n^{(p)} = sA_n^{(p)} + \alpha_n n^{p-\rho}$, where $\alpha_n \to 0$ and $\sum n^{-1}\alpha_n$ is convergent, then $\sum n^\rho a_n$ is summable to a sum s' for which a formula is given. Analogous but different criteria for summability (C, p) of $\sum n^\rho a_n$ are given for the case $\rho = -1$. *R. P. Agnew* (Ithaca, N. Y.).

Kloosterman, H. D. On the convergence of series summable (C, r) and on the magnitude of the derivatives of a function of a real variable. J. London Math. Soc. **15**, 91–96 (1940). [MF 2518]

The author establishes the formulas

$$(1) \quad h^r S_n^{(0)} = \Delta_h^r S_n^{(r)} - \sum_{\nu=n+1}^{n+h} (n+h+1-\nu) \sum_{\mu=0}^{r-1} h^{r-1-\mu} \Delta_h^\mu S_\nu^{(\mu-1)},$$

$$(2) \quad h^r f^{(r)}(x) = \Delta_h^r f(x) - \tfrac{1}{2} r h^{r+1} f^{(r+1)}(\xi), \qquad x < \xi < x + rh,$$

where

$$S_n^{(0)} = a_1 + a_2 + \cdots + a_n,$$

$$S_n^{(r)} = S_1^{(r-1)} + S_2^{(r-1)} + \cdots + S_n^{(r-1)}, \quad r = 1, 2, \cdots,$$

the rth order difference operator of span h is denoted by Δ_h^r, and $\Delta_h^0 S_n^{-1}$ is to be interpreted as a_n. Among the results which he establishes as applications of these formulas are the following: (i) If the series $\sum a_n$ is summable (C, r) to

zero and $a_n > -Kn^{-\lambda}$, where K is a positive number independent of n, and $-r<\lambda\leqq 1$, then $S_n^{(0)}=o(n^{(1-\lambda)r/(r+1)})$. (ii) If the function $f(x)$, defined for $x>0$, has derivatives up to the $(r+1)$th order, and as $x\to\infty$, $f(x)=o(x^r)$, $f^{(r+1)}(x)>-Kx^{-\lambda}$, where K is a positive number independent of x and $-r<\lambda\leqq 1$, then $f^{(r)}(x)=o(x^{(1-\lambda)r/(r+1)})$. The author refers to three notes in the Journal of the London Mathematical Society, vol. **3** (1928), in which L. J. Mordell has given simplified proofs of some theorems of Hardy and Littlewood as well as generalizations of these theorems, and he asserts that all the results of Mordell follow immediately from formulas (1) and (2). *D. Moskovitz.*

Birindelli, Carlo. Sui metodi di Gronwall per la sommazione delle serie. Ann. Scuola Norm. Super. Pisa (2) **8**, 241–270 (1939). [MF 2040]

The method of summability (f, g) introduced by T. H. Gronwall [Ann. of Math. (2) **33**, 101–117 (1932)] is based upon two analytic functions $f(w)$ and $g(w)=\sum_0^\infty b_n w^n$. The transforms $\{U_n\}$ of the series $\sum u_\nu$ are defined by

$$g(w)\sum_0^\infty u_\nu z^\nu=\sum_0^\infty b_n U_n w^n, \qquad z=f(w).$$

The functions $f(w)$ and $g(w)$ satisfy certain specified mapping and regularity conditions. The author considers the case in which $w=F(z)$ is given instead of $z=f(w)$. In particular, he takes $w=za^{1-z}$, $a\geqq 1$, $g(w)=1/(1-z)$. The explicit expression of U_n is simple in this case and the method is suitable for analytic continuation of a power series in as much as every fixed point in the interior of the Mittag-Leffler principal star of the function defined by the series lies within the domain of summability of the method provided a is sufficiently large. Some methods of summability introduced by N. Obrechkoff are shown to be special Gronwall methods. If $U_n^{(1)}=\sum_0^n a_{n,\nu}^{(1)}u_\nu$, $U_n^{(2)}=\sum_0^n a_{n,\nu}^{(2)}u_\nu$ are two sequences of Gronwall sums, then $U_n^{(3)}=\sum_0^\infty a_{n,\nu}^{(1)}a_{n,\nu}^{(2)}u_\nu$ defines a regular method of summation. The domain of summability of the geometric series by this method is determined. Some instances of non-regular Gronwall methods are discussed. *E. Hille* (New Haven, Conn.).

Hayashi, Gorō and Izumi, Shin-ichi. Theorems on Nörlund's method of summation. I and II. Tôhoku Math. J. **47**, 6–13, 69–73 (1940). [MF 2617]

First paper: The matrix of the inverse of Nörlund's transformation (N, q) is expressed in terms of determinants, and this result is used to obtain criteria involving determinants for relative inclusion of different Nörlund methods of summability. An inequality of Bloch-Pólya is used to estimate the determinants and to obtain simpler sufficient conditions for relative inclusion. Necessary conditions for (N, q) summability are obtained for the cases in which the sequence q is monotone or has monotone differences. Tauberian theorems of the o type are given. The last Tauberian theorem runs as follows. If (N, q) is regular, $q_n>0$, $\sum a_n$ is summable (N, q), and $na_n=o(1)$, then $\sum a_n$ converges. It is stated that this is an immediate consequence of the Tauberian theorem for Abel summability and the following theorem due to Silverman and Tamarkin: If (N, q) is regular and $q_n>0$, then Abel's method includes (N, q). It must be remarked that the Silverman-Tamarkin theorem concludes that a generalized Abel method includes (N, q) and hence that a generalization of the familiar Tauberian theorems for Abel summability is required. The required generalization is easily obtained, for if $\sum a_n$ is summable by the generalized Abel method and the Tauberian condition $na_n=o(1)$ is satisfied, then $\sum a_n$ is summable by the Abel method.

Second paper: Two theorems give conditions on sequences p_n and q_n which ensure that convergence of the sequence

$$y_n=x_n+q_n(p_n x_1+p_{n-1}x_2+\cdots+p_1 x_n)$$

implies convergence of the sequence x_n. Conditions are obtained which ensure that a Nörlund method of summability is equivalent to the Riesz method obtained by reversal of the order of the elements of the rows of the Nörlund matrix. *R. P. Agnew* (Ithaca, N. Y.).

Wall, H. S. Continued fractions and totally monotone sequences. Trans. Amer. Math. Soc. **48**, 165–184 (1940). [MF 2739]

The author discusses "correspondence" between continued fractions

$$(1) \qquad \frac{b_1|}{|1}+\frac{b_2x|}{|1}+\frac{b_3x|}{|1}+\cdots, \qquad \text{all } b_n>0,$$

and power series

$$(2) \qquad c_0-c_1x+c_2x^2\mp\cdots$$

by means of certain properties of continued fractions and some known results of Stieltjes. The main result is as follows. Theorem I. The sequence $\{c_n\}$ is totally monotonic ($\Delta^m c_n\geqq 0$; $m, n=0, 1, \cdots$) if and only if (2) has a corresponding continued fraction of the form

$$(3) \qquad \frac{g_0|}{|1}+\frac{g_1x|}{|1}+\frac{(1-g_1)g_2x|}{|1}+\frac{(1-g_2)g_3x|}{|1}+\cdots, \qquad 0\leqq g_n\leqq 1.$$

A study is made of the convergence properties of (3); also (3) is obtained for some hypergeometric series, and its properties are indicated corresponding to the moment-problem for $(-\infty, 1)$. The author further derives necessary and sufficient conditions that $f(x)=c_0-c_1x+c_2x^2\mp\cdots$ in Theorem I shall be bounded in the region $|x|<1$, in terms of (3), in terms of the moments c_n ($\sum_{n=0}^\infty c_n$ converges), and also in terms of Stieltjes integral representation ($f(x)=\int_0^1(1-u)d\varphi(u)/(1+xu)$, $\varphi(u)\uparrow$ and bounded). In closing the author shows the connection between the foregoing results and those of J. Schur concerning power series bounded in the interior of the unit-circle. *J. Shohat.*

Garabedian, H. L. and Wall, H. S. Hausdorff methods of summation and continued fractions. Trans. Amer. Math. Soc. **48**, 185–207 (1940). [MF 2740]

This paper is closely related to one by Wall [cf. the preceding review]. It uses the same notations and similar methods, based, in addition, on the work of Hausdorff concerning the moment-problem for (0, 1), and also on his matrix-methods of summation. In the first part we are given a (totally monotone) sequence $\{c_m=\int_0^1 u^m d\varphi(u);\ m=0, 1, \cdots\}$. The authors study its "regularity," that is, whether the matrix

$$(*) \qquad D(\delta_{mn}c_m)D;\ \delta_{mn}=0\,(m\neq n),\ =1\,(m=n);\ D=((-1)^n C_{mn}),$$

is regular (in the sense of Toeplitz-Silverman), and also the regularity of $\{c_0, \Delta c_0, \Delta^2 c_0, \cdots\}$. The conditions for regularity are expressed in terms of the continued fraction (3) [preceding review] and also in terms of Stieltjes integral representation. The second part deals with the $\{c_n\}$ obtained from the hypergeometric series by means of (3), which leads to a discussion of "hypergeometric" summability $(H, \alpha, \beta, \gamma)$, in particular, $(H, \alpha, 1, \gamma)$, $\gamma>\alpha>0$, by

means of a Hausdorff matrix (*). Finally the authors replace $\{c_n\}$ in the matrix $\Delta = (\Delta^m c_n)$ by some known regular sequences and discuss the resulting methods of summation. Application is made to the analytic continuation of a power series outside its circle of convergence. *J. Shohat.*

Nigam, Tapeshwari Prasad. On γ-transformations of series. Proc. Edinburgh Math. Soc. (2) **6**, 123–127 (1940). [MF 2842]

The first theorem gives necessary and sufficient conditions on functions $g_k(a)$ that a regular method Γ of summability determined by the transformation $\gamma(a) = \sum g_k(a) c_k$ be such that each series $\sum c_n$ having bounded partial sums is summable Γ. The theorem is illusory in the sense that the conditions obtained are inconsistent with regularity of Γ; but of course the theorem is in agreement with the fact that no regular method Γ can evaluate all bounded sequences. The second theorem shows that certain divergent series are summable to different values by different regular methods. [For a more complete discussion of the subject, see Agnew, Ann. of Math. (2) **32**, 715–722 (1931); the theorem is also contained in a recent result of Darevsky; cf. the following review.] The remainder of the paper deals with inclusion and equivalence relations between series-to-sequence transformations, the results being analogous to the well-known results involving sequence-to-sequence transformations. *R. P. Agnew* (Ithaca, N. Y.).

Darevsky, W. Sur certains problèmes de la théorie des séries divergentes. Rec. Math. [Mat. Sbornik] N.S. **7** (49), 549–590 (1940). (Russian. French summary) [MF 2806]

In the first part of this paper the author discusses the question of how arbitrary the sum of a real divergent series can be. His first theorem is that any divergent series can be summed by some regular summation method to any given real number (by "regular summation method" we mean here a method which is simultaneously a matrix method and a convergence factor method, and which sums all convergent series to the right values). Much stronger theorems were proved by Agnew [Ann. of Math. (2) **32**, 715–722 (1931)]. A slight extension then follows easily, namely that for any divergent series there are regular "functional matrix" summation processes [Darevsky, Rec. Math. [Mat. Sbornik] **41**, 458–482 (1934)] summing the series to $+\infty$ and $-\infty$. The author's third theorem is that any divergent series can be summed to any given number between the upper and lower bounds of its partial sums by some regular summation method which does not enlarge the limits of indetermination of any series. [A somewhat stronger theorem in Dienes, The Taylor Series, London, 1931, p. 391. The author's statement in the summary does not seem to correspond to his text; we should apparently read, "Soit M l'ensemble" instead of "Soit M un ensemble" at the beginning of Theorem III, p. 588.]

The author devotes the second and longer part of the paper to the problem of finding classes of mutually consistent regular summation methods, that is, classes such that whenever two methods of a class apply to the same series, they give the same sum. With the idea that the preservation of some of the simple properties of convergence may be the deciding factor, the author introduces eight conditions of varying restrictiveness, of which a typical one is (3): If $\sum a_n$ and $\sum a_n'$ are both summable, where $\sum a_n'$ is obtained by inserting the terms of a convergent series (in their original order) among the terms of $\sum a_n$, then the sum of the second is the sum of the first plus the sum of the added series. The author's results are almost entirely negative. He shows that for each of two of his conditions [one being (3)], there exists a pair of summation methods satisfying the condition but summing some one series to different values. Thus these conditions cannot determine classes of mutually consistent methods. It is shown that there are methods, not equivalent to convergence, satisfying three of the other conditions. Thus it is possible that these may lead to solutions of the author's problem, though he adduces no further evidence. The two most restrictive conditions are shown to be so restrictive that a summation method possessing them is necessarily equivalent to convergence. This follows from the following theorem which the author proves [extending a result of Steinhaus, Prace Mat.-Fiz. **22**, 121–134 (1911); Dienes, op. cit., p. 392]: Given a regular matrix summation method and a divergent series, a series not summable by the given method can be constructed by inserting zeros among the terms of the given series. In Steinhaus's construction the given series has all its terms equal to unity. *R. P. Boas, Jr.* (Durham, N. C.).

Hurwitz, Henry, Jr. Total regularity of general transformations. Bull. Amer. Math. Soc. **46**, 833–837 (1940). [MF 2930]

A transformation $y(t) = \sum a_k(t) x_k$ is totally regular if it is regular and also $y(t) \to \infty$ whenever $x_k \to \infty$. It was shown by W. A. Hurwitz that a real regular triangular-matrix transformation $y_n = \sum_{k=1}^{n} a_{nk} x_k$ is totally regular if and only if (i) there is an index K such that $a_{nk} \geqq 0$ when $k \geqq K$. For total regularity of real regular square-matrix transformations $y_n = \sum_{k=1}^{\infty} a_{nk} x_k$, the condition (i) is sufficient, but examples show that (i) is not necessary. Henry Hurwitz obtains four theorems, each of which characterizes the real regular transformations $y(t) = \sum a_k(t) x_k$ which are totally regular. Each theorem contains two conditions; the first is that, for each sufficiently advanced t, there shall be only a finite set of values of k for which $a_k(t) < 0$; the second is a more subtle condition involving "guards."

R. P. Agnew (Ithaca, N. Y.).

Amerio, Luigi. Un metodo di sommazione per le serie di potenze e sua applicazione alla trasformazione di Laplace. Ann. Scuola Norm. Super. Pisa (2) **8**, 167–180 (1939). [MF 2035]

Let l_1 represent the loop of the curve in the z-plane defined by $|ze^{1-z}| = 1$; l_1 is a convex curve about the origin, symmetric with respect to the x-axis and with a single 90° vertex at $z=1$. Let l_λ represent the curve described by λz as z describes l_1. The power series (1) $f(z) = \sum_0^\infty a_m z^m$ with a positive radius of convergence is said to be summable (S) at the point z provided the sequence of polynomials (2) $k_n(z) = \sum_{m=0}^{n} \binom{n}{m} m! n^{-m} a_m z^m$ tends to a limit as $n \to \infty$. It is shown that the series (1) is summable (S) at $z = \lambda$ to the value $f(\lambda)$ provided $f(z)$ is regular inside and on the curve l_λ, and not summable if $f(z)$ has a singular point inside l_λ. This result allows one to specify completely the domain of summability (S) of (1) and also leads to a criterion for regularity of points on the circle of convergence of (1). The application to a Laplace transform (3) $f(p) = \int_0^\infty e^{-pt} F(t) dt$, $\Re p > 0$, is as follows. If λ and p are such that $\Re\lambda > 0$, $\Re\lambda \geqq |\Im\lambda|$, $\Re p > -\Re\lambda$, the summation method (S) is applied to the power series $f(p+\lambda+z) = \sum_{m=0}^{\infty} z^m f^{(m)}(p+\lambda)/m!$ at the point $z = -\lambda$. The approximating polynomials (2) are now expressible by an

integral, the summation process leading to the relation

$$(4) \qquad \lim_{n\to\infty} \int_0^\infty e^{-(p+\lambda)t}\left(1+\frac{\lambda t}{n}\right)^n F(t)dt = f(p).$$

Beyond the domain $\Re p>0$ of convergence of (3), (4) remains valid at points of regularity of $f(p)$ on the line $\Re p=0$ and also further to the left in a certain region completely specified in terms of the singularities of $f(p)$. It further implies criteria for regularity of points on $\Re p=0$ for the statement of which we must refer to the paper.

I. J. Schoenberg (Waterville, Me.).

Kloosterman, H. D. Limitierungsumkehrsätze mit Lückenbedingungen für das C-Verfahren. Math. Z. **46**, 375–379 (1940). [MF 2405]

The author gives a considerably simplified proof as well as an extension of certain results of Meyer-König insofar as Cesàro summability is concerned. [Cf. Math. Z. **45**, 447–478 (1939); these Rev. **1**, 11.] *N. Levinson.*

Pitt, H. R. General Tauberian theorems. II. J. London Math. Soc. **15**, 97–112 (1940). [MF 2519]

The following extension of Wiener's general Tauberian theorem was proved in recent papers by the author [Amer. J. Math. **60**, 532–534 (1938); Proc. London Math. Soc. (2) **45**, 243–288 (1938)]. Theorem 1. Hypothesis. (I) $k_1(x)$, $k_2(x)$ are measurable and absolutely integrable in $(-\infty, \infty)$; (II) $K_1(u)=\int_{-\infty}^{\infty}e^{-iux}k_1(x)dx\neq 0$ $(-\infty<u<\infty)$; (III) $|f(x)|\leqq M$, (IV) $g_1(x)=\int_{-\infty}^{\infty}k_1(x-y)f(y)dy$, $g_2(x)=\int_{-\infty}^{\infty}k_2(x-y)f(y)dy$. Conclusion. (A) $G_2\leqq\Psi(G_1)$, where, if g_1 is bounded, we write $G_1=\overline{\lim}_{x\to\infty}\, g_1(x)$, and similarly for the other letters, and where $\Psi(\xi)$ depends only on $k_1(x)$, $k_2(x)$ and M, and $\Psi(0)=\Psi(+0)=0$. (B) If we have the further condition (V) $\overline{\lim}_{x\to\infty}|f(x+\epsilon)-f(x)|=\delta(\epsilon)$, $\lim_{\epsilon\to 0}\delta(\epsilon)=0$, then $F\leqq\Phi(G_1)$, where $\Phi(\xi)$ depends only on $k_1(x)$ and $\delta(\epsilon)$ (not on M), and $\Phi(0)=\Phi(+0)=0$.

Previously only case (B) was stated, but the author now shows that (A) follows from it very easily. The object of this paper is to show that, even if $K_1(u)$ vanishes for certain values of u, Theorem 1 remains true if we have extra conditions on $f(x)$ and $k_2(x)$. Main results are generalizations of Theorem 1 in which the extra condition, instead of on $k_2(x)$, is on $f(x)$. When $K_1(u)\neq 0$ except at α, the kind of condition which is required to make Theorem 1 remain true is one which excludes functions $f(x)$ behaving in any way like $e^{i\alpha x}$, that is, which make the harmonic component of $f(x)$ with frequency near to α small. The most natural way of stating this condition is in terms of the harmonic representation of $f(x)$ defined by Wiener [Acta Math. **55**, 117–258 (1930)], and extended by the author [Proc. London Math. Soc. (2) **46**, 1–18 (1939); these Rev. **1**, 139].

S. Ikehara (Osaka).

Agnew, Ralph Palmer. On Tauberian theorems for double series. Amer. J. Math. **62**, 666–672 (1940). [MF 2463]

A sufficient condition for a Tauberian theorem in the case of a Cesàro summable double series $\sum_{m,n=1}^{\infty}a_{m,n}$ is $a_{m,n}<K/(m^2+n^2)$. In answer to a question of Knopp as to whether the weaker condition $a_{m,n}<K/mn$ suffices as a Tauberian condition, the author shows that it does not, and in fact that even stronger conditions of this type are insufficient. *N. Levinson* (Cambridge, Mass.).

Nigam, Tapeshwari Prasad. Summability of multiple series. Proc. London Math. Soc. (2) **46**, 249–269 (1940). [MF 2359]

Let

$$s_{kl}=\sum_{i,j=1}^{k,l}c_{ij}$$

denote the sequence of partial sums of a double series $\sum c_{ij}$. The series $\sum c_{kl}$ is convergent (c) if s_{kl} has a limit as $k, l\to\infty$; is boundedly convergent (bc) if c and s_{kl} is bounded; is ultimately regularly convergent (urc) if c and if there exists an index Q such that $\lim_{k\to\infty}s_{kl}$ and $\lim_{l\to\infty}s_{kl}$ exist, respectively, for $l>Q$ and $k>Q$; is regularly convergent (rc) if Q can be taken to be 0; and is boundedly ultimately regularly convergent (burc) if both bc and urc. A matrix g_{kl} whose elements are complex-valued functions of a and b determines a series-to-function transformation

$$\gamma(a, b)=\sum_{k,l}^{\infty}g_{kl}(a, b)c_{kl}$$

by means of which $\sum c_{kl}$ is summable to s if $\gamma(a, b)\to s$ as $a, b\to\infty$. Twenty-five theorems characterize the matrices such that each c (or bc or rc or urc or burc) series $\sum c_{kl}$ has a transform $\gamma(a, b)$ which is c (or bc or rc or urc or burc). Further theorems characterize the matrices for which the additional condition $\lim\gamma(a, b)=\sum c_{k,l}$ is satisfied. Corresponding problems for sequence-to-sequence transformations

$$\sigma_{i,j}=\sum_{k,l=1}^{\infty}a_{ijkl}s_{kl}$$

have been solved by T. Kojima [Tôhoku Math. J. **21**, 3–14 (1922)], G. M. Robinson [Trans. Amer. Math. Soc. **28**, 50–73 (1926)] and H. J. Hamilton [Duke Math. J. **2**, 29–60 (1936)], and by other authors cited by Hamilton.

R. P. Agnew (Ithaca, N. Y.).

Fourier Series and Integrals, Theory of Approximation

Geronimus, J. Sur un problème-minimum. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 224–226 (1939). [MF 2907]

By means of a certain Fourier expansion the following result is established:

$$\int_0^{2\pi}\left|\frac{C(\theta)}{q(z)}\right|d\theta\geqq 4, \qquad z=e^{i\theta},$$

equality attained. Here

$$C(\theta)=\Re\{\sum_{s=0}^{n}(\alpha_s+i\beta_s)e^{i(n-s)\theta}\}$$

belongs to the class of trigonometric polynomials, of order n, subject to the condition $\alpha_0\lambda_0+\beta_0\mu_0=1$; $q(z)$ is a given polynomial, of degree $s\leqq 2n-1$, properly specified. Application is made to polynomials on $(-1, 1)$. *J. Shohat.*

Schultze, Johann Friedrich. Über Kosinuspolynome und die Nullstellen von Polynomen. Jber. Deutsch. Math. Verein. **50**, 35–43 (1940). [MF 2772]

The purpose of the paper is to find sufficient conditions which, if fulfilled by the coefficients $\{a_\nu\}$, are such that $c_n(\varphi)=a_0/2+a_1\cos\varphi+\cdots+a_n\cos n\varphi\geqq 0$ for every φ. Such conditions are obtained by identifying $c_n(\varphi)$ with a sum of the form $\sum_0^n A_\nu P_\nu$, the $\{A_\nu\}$ being numerical factors and P_ν

being a non-negative cosine polynomial of order ν, such as $P_\nu=\cos^{2\nu}\varphi/2$, $P_\nu=\frac{1}{2}|B_0+B_1e^{i\varphi}+\cdots+B_\nu e^{i\nu\varphi}|^2$. The $\{A_\nu\}$ are expressed as functions of the $\{a_\nu\}$ by means of determinants, and sufficient conditions are obtained by writing $A_\nu\geqq 0$ $(\nu=0, 1, \cdots, n)$. *R. Salem* (Montreal, P. Q.).

Lozinski, S. Über trigonometrische Interpolation. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 229–248 (1940). (Russian. German summary) [MF 2738]

Marcinkiewicz has proved [Studia Math. **6**, 1–17, 67–81 (1936); see also Marcinkiewicz and Zygmund, Fund. Math. **28**, 131–166 (1937)] that if $S(x)$ is an arbitrary trigonometric polynomial of order n, and if

$$(*)\qquad x_n^{(k)}=\frac{2\pi k}{2n+1},\quad k=0, 1, \cdots, 2n; 1\leqq p<\infty,$$

then

$$\left(\frac{2\pi}{2n+1}\sum_{k=0}^{2n}|S(x_k)|^p\right)^{1/p}\leqq A\int_0^{2\pi}|S(x)|^p dx,$$

where A is an absolute constant. Generalizing the result the author shows that if $\varphi(n)$ is any nondecreasing convex function such that $\varphi(0)=0$, then

$$\frac{2\pi}{2n+1}\sum_{k=0}^{2n}\varphi(|S(x_k)|)\leqq\int_0^{2\pi}\varphi(4|S(x)|)dx.$$

Moreover the author extends to interpolating trigonometric polynomials some Parseval relations known for Fourier series, and generalizes to a class of bounded functions certain theorems proved previously for continuous functions. For example, Marcinkiewicz [loc. cit.] and Erdös-Feldheim [C. R. Acad. Sci. Paris **203**, 913–915 (1936)] proved that if $U_n(x;f)$ is the trigonometric polynomial of order n coinciding with the function $f(x)$ at the points (*), then

$$(\substack{*\\*})\qquad \int_0^{2\pi}|f(x)-U_n(x;f)|^p dx\rightarrow 0$$

for every fixed $p>0$, provided that f is continuous [even slightly more is true then, namely that

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\lambda|f(x)-U_n(x;f)|dx\rightarrow 1$$

for every $\lambda>0$; see Marcinkiewicz and Zygmund, loc. cit.]. The author shows that $(\substack{*\\*})$ is true for every bounded function with discontinuities only of the first kind.
A. Zygmund (South Hadley, Mass.).

Cameron, R. H. Quadratures involving trigonometric sums. J. Math. Phys. Mass. Inst. Tech. **19**, 161–166 (1940). [MF 2640]

Let $Y'(x)=P(x)Y(x)+Q(x)$, where $P(x)$ and $Q(x)$ belong either to the class $\mathfrak{U}$ of uniformly almost periodic functions or to a class $\mathfrak{A}_\theta$ of absolutely convergent Fourier-Stieltjes transforms for which a certain norm is finite. [Cf., for the definition of the norms, Trans. Amer. Math. Soc. **46**, 97–109 (1939); these Rev. **1**, 13, where, however, $df(y)$ should be replaced by $|df(y)|$ in the first displayed formula.] Assuming the real part of the average of $P(x)$ to be different from zero, the author proves that the differential equation has one and only one solution $y(x)$ bounded on $-\infty<x<\infty$, and that $y(x)$ belongs to the same class $\mathfrak{U}$ or $\mathfrak{A}_\theta$ as the coefficients do. *E. Hille* (New Haven, Conn.).

✶Salem, Raphaël. Essais sur les séries trigonométriques. Thèse presentée à la Faculté des Sciences de l'Université de Paris. Hermann et Cie., Paris, 1940. 87 pp.

The present dissertation (scheduled to appear also in the Actualités Scientifiques) contains results published by the author (sometimes only with indications of proofs) in the C. R. Acad. Sci. Paris within the last twelve years. In Chapter I we find the study of the asymptotic behavior in the neighborhood of the point $x=0$ of the functions represented by the series

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx,\quad \sum_{n=1}^{\infty}b_n\sin nx,$$

where the coefficients a_n, b_n tend monotonically to 0 and possess certain additional properties. Ch. II gives a necessary and sufficient condition for a trigonometric series to be a Fourier series. In Ch. III we have a proof of the theorem that, if in the trigonometric polynomial

$$P(x)=r_1\cos(x-\alpha_1)+r_2\cos(2x-\alpha_2)+\cdots+r_n\cos(nx-\alpha_n)$$

the coefficients r_n are given, the phases α_γ may be so chosen that the maximum M of $|P|$ satisfies the inequality

$$M\leqq C\sqrt{\log n}\,(r_1^2+r_2^2+\cdots+r_n^2)^{\frac{1}{2}},$$

where C is an absolute constant. The main result of Ch. IV is that, if the series $\sum r_n^2$ converges and the sequence $1/r_n$ is concave, then it is possible to choose real numbers α_γ in such a way that the series

$$\sum_{\gamma=1}^{\infty}r_\gamma e^{2\pi i(\gamma x-\alpha_\gamma)}$$

converges uniformly (no matter how slow is the convergence of the series $\sum r_n^2$). In Ch. V the author studies the phenomena of absolute convergence of Fourier series. He proves that, if $\omega(\delta)$ denotes the modulus of continuity of a function f, and if

$$f(x)\sim\sum_{\gamma=1}^{\infty}r_\gamma\cos(\gamma x-\alpha_\gamma),$$

then, for every function $F(u)$ positive, increasing and concave, the series $\sum F(n_n^2)$ converges if $\sum F(n^{-1}\omega^2(1/n))$ is finite (this generalizes well-known results of S. Bernstein and Szász). The result cannot be improved. Ch. VI gives contributions to the uniform convergence of Fourier series and contains an interesting criterion for the uniform convergence of Fourier series generalizing both the Dini and the Jordan tests. In Ch. VII the author applies to the theory of Fourier series two methods of summation defined respectively as

$$\lim_{s\rightarrow+0}\left\{\tfrac{1}{2}a_0+\sum_{\gamma=1}^{\infty}\frac{a_\gamma\cos\gamma x+b_\gamma\sin\gamma x}{1+s\log\gamma}\right\},$$

$$\lim_{s\rightarrow+0}\left\{\tfrac{1}{2}a_0+\sum_{\gamma=1}^{\infty}\frac{a_\gamma\cos\gamma x+b_\gamma\sin\gamma x}{1+s\sqrt{\log\gamma}}\right\}.$$

In Ch. VIII it is shown that, if $\omega(\delta)$ is the integral modulus of continuity of a function $f\epsilon L$ and if the sum $\sum\omega(n_k^{-1})\log\omega(n_{n_k}^{-1})$ is finite, then the partial sums $s_{n_k}(x)$ of the Fourier series of f converge almost everywhere. The final Ch. IX contains the result that, if the series $\sum r_n$ with positive terms is divergent, then it is possible to find real numbers α_γ in such a way that the partial sums of the series $\sum r_\gamma e^{2\pi i(\gamma x-\alpha_\gamma)}$ are unbounded at an everywhere dense set of points. *A. Zygmund* (South Hadley, Mass.).

Fedoroff, W. S. Sur les coefficients de Fourier. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 41–56 (1940). (Russian. French summary) [MF 3187]

Let $f(z)$ be a function defined and continuous (but not necessarily analytic) in a domain D of the plane of the complex variable $z=x+iy$. Let $\Gamma(z, r)$ denote the circumference with center z and radius r, and let (z, r) denote the circle limited by this circumference. If $\Gamma(z, r)$ is contained in D we may consider the Fourier coefficients of the function $f(z+re^{i\varphi})$, where $0\leqq\varphi\leqq2\pi$. The numbers a_n, b_n depend not only on n but also on z and r, so that $a_n=a_n(z, r)$, $b_n=b_n(z, r)$. The problem investigated by the author is to characterize the function f by the functions $a_n(z, r)$ and $b_n(z, r)$ for some fixed n. We quote a few results. Let

$$L_n(z, r)=\pi ir^n[a_n(z, r)+ib_n(z, r)]=\oint_{\Gamma(z,r)} f(t)(t-z)^{n-1}dt,$$

$$I_n(z, r)=-i\int_0^r L_n(z, \rho)d\rho=\int_{(z,r)} f(t)(t-z)^n d\sigma.$$

Then (i) a necessary and sufficient condition that $I_n(z, r)=0$ for every (z, r) is that

$$(*)\qquad f(z)=\sum_{k=0}^{n-1} h_k(z)x^k,$$

where the $h_k(z)$ are regular in D. (ii) A necessary and sufficient condition for the validity of (*), where $h_{n-1}(z)$ does not vanish identically, is that $I_n=0$ for every (z, r) and that $I_{n-1}(z_0, r_0)$ for some (z_0, r_0). (iii) A necessary and sufficient condition that the function $f(z)$, real and continuous in D, should be a polynomial in x, y in the domain D is that there should exist an integer n such that for any domain Δ interior to D the expressions $r^{-n}a_n(z, r)$, $r^{-n}b_n(z, r)$ should tend uniformly to 0 in Δ as $r\to0$. *A. Zygmund.*

Foà, Alberto. Sulla sommabilità assoluta $|C, \alpha|$ delle serie di Fourier di una funzione sommabile L^p con $p>1$. Boll. Un. Mat. Ital. (2) **2**, 325–332 (1940). [MF 2985]

It is proved that the $|C, \alpha|$ summability of the Fourier series of a function $f\epsilon L^p$, where $p>1$, is a local property for $\alpha>1/p$, but not for $\alpha\leqq1/p$. [A series is said to be summable $|C, \alpha|$ if the Cesàro means σ_n^α of the series satisfy the condition $|\sigma_1^\alpha-\sigma_0^\alpha|+|\sigma_2^\alpha-\sigma_1^\alpha|+\cdots<\infty$.]
A. Zygmund (South Hadley, Mass.).

Izumi, Shin-ichi and Kawata, Tatsuo. Notes on Fourier series (XI). Inequality theorem in the strong summability. Tôhoku Math. J. **47**, 14–17 (1940). [MF 2618]

Let $s_n(x)$ and $\sigma_n(x)$ denote the partial sums and the first arithmetic means of the Fourier series of a function $f(x)$ of the class L^k, where $k\geqq1$. It is known [Zygmund, Fund. Math. **30**, 170–196 (1938)] that, if $k>1$, then the function

$$\sum_{\gamma=0}^{\infty}\frac{|s_\gamma(x)-\sigma_\gamma(x)|^k}{\gamma+1},$$

and so also the function

$$\operatorname{Max}_n \frac{1}{n+1}\sum_{\gamma=0}^{n}|s_\gamma(x)-\sigma_\gamma(x)|^k,$$

belongs to L^k. In the present paper the authors show by examples that both facts are no longer true for $k=1$.
A. Zygmund (South Hadley, Mass.).

Kawata, Tatsuo. A relation between the theories of Fourier series and Fourier transforms. Proc. Imp. Acad. Tokyo **16**, 255–261 (1940). [MF 2942]

Proofs are given of the following two theorems. (i) Suppose that a function $f(x)$ of the class $L^p(-\infty, \infty)(p>1)$ has a Fourier transform $F(t)\epsilon L^q(-\infty, \infty)$ for some $q\geqq1$. Let c_n be the Fourier coefficients of the function $\varphi(t)$ which is of period $2R$ and coincides with $F(t)$ over the interval $(-R, R)$. Then

$$\sum_{n=-\infty}^{\infty}|c_n|^p\leqq\frac{A_p}{R^{p-1}}\int_{-\infty}^{\infty}|f(x)|^p dx,$$

where A_p depends only on p. (ii) Suppose that $\varphi(t)$ is any function of the class $L(-R, R)$ and of period 2π, and that $f(x)$ is the Fourier transform of the function $F(t)$ equal to $\varphi(t)$ in $(-R, R)$ and to 0 otherwise. Then the Fourier coefficients c_n of the function $\varphi(t)$ satisfy the inequality

$$\int_{-\infty}^{\infty}|f(x)|^p dx\leqq A_p R^{p-1}\sum_{n=-\infty}^{\infty}|c_n|^p,$$

where again A_p depends on p only. *A. Zygmund.*

Natanson, I. P. Sur un procédé de sommation des intégrales de Fourier. Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 313–320 (1940). (Russian. French summary) [MF 2793]

Let $f(t)\epsilon L(-\infty, +\infty)$ and let

$$S_\lambda(x)=\frac{1}{\pi}\int_0^\lambda du\int_{-\infty}^{+\infty} f(t)\cos u(t-x)dt,$$

$$B_\lambda(x)=\frac{1}{2}\left[S_\lambda(x)+S_\lambda\left(x+\frac{\pi}{\lambda}\right)\right].$$

The author studies the behavior of the expression $B_\lambda(x)$ for $\lambda\to\infty$. This is an extension (with a minor change) to Fourier integrals of a method applied previously by Rogosinski [Math. Ann. **95**, 110–134 (1925)] to Fourier series. It is shown that (i) at every point x_0 where

$$\lim_{h\to0}\frac{1}{h}\int_0^h|f(x_0+t)-f(x_0)|dt=0,$$

we have $B_\lambda(x_0)\to f(x_0)$; (ii) if $f(t)\epsilon\mathrm{Lip}\,\alpha$ $(0<\alpha<1)$, then $B_\lambda(x)-f(x)=O(\lambda^{-\alpha})$, (iii) if $f(t)\epsilon\mathrm{Lip}\,1$, then $B_\lambda(x)-f(x)=O(\lambda^{-1}\log\lambda)$; (iv) if $f'(x_0\pm0)$ exist and are finite, then $B_\lambda(x_0)-f(x_0)=o(\lambda^{-1}\log\lambda)$. *A. Zygmund.*

Levinson, N. Restrictions imposed by certain functions on their Fourier transforms. Duke Math. J. **6**, 722–731 (1940). [MF 2725]

This is a contribution to the theory which has developed from Wiener's remark that a pair of Fourier transforms cannot both be very small. The author's principal theorem is that if $f(x)\epsilon L(-\infty, \infty)$ and $g(x)$ is its Fourier transform, if $f(x)=O(e^{-h(x)})$ and $g(x)=O(e^{-l(x)})$ as $x\to+\infty$, if $h(x)$ is positive, increasing, with a second derivative having a positive lower bound, and $H(x)$ is the inverse of $h'(x)$, then if

$$\int_1^x\left(\frac{1}{y^2}-\frac{1}{x^2}\right)l(y)dy-\pi H(x)\to\infty$$

as $x\to\infty$, $f(x)=0$ almost everywhere. This generalizes older results of Hardy [J. London Math. Soc. **8**, 227–231 (1933)], Morgan [J. London Math. Soc. **9**, 187–192 (1934)] and Ingham [Ibid., 29–32 (1934)], both in requiring $f(x)$ and

$g(x)$ to approach zero only as $x\to\infty$ from one side, and in allowing greater variety in the rate of growth of the function $h(x)$. The constant π in the theorem is shown by an example to be best possible when $h(x)$ grows as fast as e^x. When $h(x)$ grows like a power of x, however, the constant is not best possible; in this case the author obtains further results comparable with those of Hardy and Morgan; they are a little less favorable in the rate of growth assumed for $h(x)$, but are stronger in requiring restrictions of $f(x)$ and $g(x)$ only as $x\to+\infty$ instead of as $|x|\to\infty$. The proofs depend on a well-known theorem of Carleman concerning the growth of functions analytic in a half-plane [Titchmarsh, The Theory of Functions, 1932, p. 130].

The author also studies the case where $f(x)$ approaches zero very rapidly on one side of a finite point, while $g(x)$ is small as $x\to\infty$ from one side. The problem is made to depend, by a suitable transformation, on the theorem stated above. The case where $f(x)$ is assumed to vanish over an interval, previously treated by the author [Proc. London Math. Soc. (2) **41**, 393–407 (1936)], appears as a limiting case. *R. P. Boas, Jr.* (Durham, N. C.).

Haviland, E. K. On an asymptotic formula for the Fourier transforms of distributions on certain curves. Amer. J. Math. **62**, 655–665 (1940). [MF 2462]

The curves mentioned in the title are Jordan curves with the parametric representation $x=x(\varphi)$, $y=y(\varphi)$ $(0\leqq\varphi<2\pi)$, where $x''(\varphi)$, $y''(\varphi)$ exist and are of bounded variation, and the function $h(\varphi,\psi)=x(\varphi)\cos\psi+y(\varphi)\sin\psi$ $(0\leqq\psi<2\pi)$ has the property that $\partial h/\partial\varphi$ and $\partial^2h/\partial\varphi^2$ have, for every ψ, n zeros apiece in $0\leqq\varphi<2\pi$, all the zeros being simple. These curves generalize the convex curves previously treated, which had $n=2$ and were restricted by additional conditions of smoothness [Haviland and Wintner, Duke Math. J. **2**, 712–721 (1936)]. The Fourier transform of the distribution function associated with one of the curves can be put into the form

$$\Lambda(r,\psi)=\frac{1}{2\pi}\int_0^{2\pi}\exp\,[irh(\varphi,\psi)]d\varphi.$$

An asymptotic formula for $\Lambda(r,\psi)$ as $r\to\infty$ is obtained by transforming Λ in such a way that the elementary method of estimation used by Hartman can be applied [Amer. J. Math. **62**, 115–121 (1940); these Rev. **1**, 140]. The formula contains contributions from all the zeros of $\partial h/\partial\varphi$; it is of the same general form as the formula of Haviland and Wintner for convex curves. The remainder term, however, is $o(r^{-1})$ instead of $O(r^{-1})$. *R. P. Boas, Jr.*

Kawata, Tatsuo. On symmetric Bernoulli convolutions. Amer. J. Math. **62**, 792–794 (1940). [MF 2880]

Refining some known results of Wintner [Amer. J. Math. **57**, 839 (1935); ibid. **56**, 659–663 (1934), and Bull. Amer. Math. Soc. **41**, 137–138 (1935)], the author proves that, if $p(t)>0$ is increasing and such that

$$\int_1^\infty\frac{p(t)}{t^3}dt<\infty,$$

a sequence $\{b_k\}$ $(\sum b_k^2<\infty)$ can be chosen so that

$$\prod_{k=1}^\infty\cos b_kt=O(\exp\,(-p(|t|))$$

as $t\to\pm\infty$. *M. Kac* (Ithaca, N. Y.).

Potter, H. S. A. The mean values of certain Dirichlet series, I. Proc. London Math. Soc. (2) **46**, 467–478 (1940). [MF 2825]

Let $f(s)=\sum a_nl_n^{-s}$ be a Dirichlet series; let σ_a be the smallest number such that f is of finite order for $\sigma\geqq\sigma_a+\epsilon$ and all its singularities in $\sigma\geqq\sigma_a+\epsilon$ are contained in a rectangle of finite area, and let $\lambda(\sigma)$ be the mean order function of f. The author proves the following theorem: If (i) $\sum|a_n|^2=O(x^{\alpha+\epsilon})$, where the sum is extended over $l_n\leqq x$, (ii) for sufficiently large n, $\log\,(l_n/l_{n-1})>l_n^{-(l+\epsilon)}$, $l\geqq0$, (iii) $q=L-(L-\eta/\lambda(\eta))$, where $L=\frac{1}{2}(\alpha+l)$ and $\eta>\sigma_a$, then

$$\int_{-T}^{+T}|f(\sigma+it)|^2dt\sim2T\sum|a_n|^2l_n^{-2\sigma}$$

for $\sigma>q$. F. Carlson [Ark. Mat. Astr. Fys. **19**, no. 25 (1926)] proved a similar theorem replacing (i) by $a_n=O(l_n^{k+\epsilon})$. The present author discusses furthermore applications of his result, in particular to Hecke's Dirichlet series with signature $\{x,k,j\}$. *F. Bohnenblust* (Princeton, N. J.).

Bochner, S. Hadamard's theorem for Dirichlet series. Ann. of Math. (2) **41**, 711–714 (1940). [MF 3011]

After a mapping by means of $z=e^{-s}$, $s=\sigma+it$, the star of a function analytic in the neighborhood of the origin in the z-plane becomes a domain of the form $\varphi(t)<\sigma<\infty$, $-\infty<t<\infty$. Denote this domain by $D(\varphi)$. After such a mapping Hadamard's theorem on composition of singularities of power-series becomes: If two functions $F(s)=\sum_0^\infty a_ne^{-\lambda_ns}$, $G(s)=\sum_0^\infty b_ne^{-\lambda_ns}$, with $\lambda_n=n$, are analytic in the domains $D(\varphi)$, $D(\psi)$, respectively, then the composite function $H(s)=\sum a_nb_ne^{-\lambda_ns}$ is analytic in the composite domain $D(X)$ with $X(t)=\sup_\tau\{\varphi(\tau)+\psi(t-\tau)\}$. For Dirichlet series in general, with the exponents λ_n monotone increasing but not necessarily integers, the result is no longer true. [This problem has been investigated by Mandelbrojt, who has established comprehensive results. See V. Bernstein, Séries de Dirichlet, 1933, chap. 8.] In the present paper the author considers another generalization referring to almost periodic functions in general rather than to functions whose Dirichlet exponents form a monotone sequence. For any almost periodic function $\varphi(t)$ a function $F(s)$ is said to belong to class C_φ if it is analytic and bounded in $D(\varphi)$ and almost periodic in some half-plane $D(\alpha)$, $\alpha=$const. In terms of this class the author proves the following theorem: If $F(s)$ belongs to C_φ and $G(s)$ belongs to C_ψ, then $H(s)$ exists and belongs to $C_{X+\delta}$ for any positive constant δ. *W. T. Martin* (Princeton, N. J.).

Meyer-König, Werner. Abelsche Sätze für Dirichletsche Reihen. Math. Z. **46**, 571–590 (1940). [MF 2783]

If $\sum a_n=s$ (where $\sum$ indicates summation for $0\leqq n$) and if $f(z)=\sum a_nz^n$, then $f(z)$ is of course analytic in the unit circle and it is well known that $\lim f(z)=s$ as $z\to1$ and $|\text{am}\,(1-z)|\leqq\alpha<\pi/2$. However $\lim f(z)$ need not exist if $z\to1$ along an arbitrary path inside the unit circle. If the path along which $z\to1$ comes into very close contact with $|z|=1$, then it is necessary to add certain restrictions on the size of a_n before it is known that $\lim f(z)=s$ along such a path. These results are known for power series. The author extends them to Dirichlet series and Laplace integrals. He also shows that in a certain direction his results are best possible. An example of the results obtained is the following theorem: If the Dirichlet series $\sum a_ne^{-\lambda_ns}$ converges for $s=0$ and if, for some $\theta>0$,

$$\sum_{x\leqq\lambda_n\leqq x^{1+\theta}}|a_n|\to0 \qquad \text{as } x\to\infty,$$

then the Dirichlet series converges uniformly for every α, $0<\alpha<1$, and every positive c in the region $(0\leqq\sigma\leqq1;\ |t|\leqq c\sigma^{\alpha})$, where $s=\sigma+it$. *N. Levinson.*

Košljakov, N. S. Some formulae for the functions $\zeta(s)$ and $\zeta_\Omega(s)$. C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 567–569 (1939). [MF 2077]

The author proves:

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{n^{-s}}{1+(n/x)^2}-\frac{\pi x^{1-s}}{2\sin(\pi(1-s)/2)}$$
$$+2x^{1-s}\int_0^\infty\left\{\sum_{n=1}^{\infty}\frac{n^{s-1}}{1+(n/x)^2}-\frac{\pi y^s}{2\sin(\pi s/2)}\right\}\frac{\cos 2\pi xy}{y^s}dy,$$
$$x>0;\ 0<\Re(s)<1.$$

Analogous formulae are obtained for other functions associated with Dirichlet's series. [Cf. A. P. Guinand, J. London Math. Soc. **14**, 97–100 (1939).] *S. Ikehara* (Osaka).

Lewitan, B. Die Verallgemeinerung der Operation der Verschiebung im Zusammenhang mit fastperiodischen Funktionen. Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 449–478 (1940). (German. Russian summary) [MF 2802]

In this paper the author generalizes the notion of an almost periodic function by generalizing the underlying idea of the operation of translation. He begins by defining the mean $M[f(t)]$ of a function $f(t)$ bounded and measurable on the whole real axis. The correspondence between functions and means is set up with the aid of the Banach limit, and hence is not unique (though a certain uniqueness property of the mean when applied to generalized a.p. functions is given at the end of the paper). The author next considers the operator $A\varphi=M_t[K(s,t)\varphi(t)]$ and shows that if the family of functions $K_t(s)=K(s,t)$ is compact then A is completely continuous, and $M_sM_t[K(s,t)]=M_tM_s[K(s,t)]$. Again, if we define the scalar product $(f,g)=M[f(t)\overline{g(t)}]$ and consider f_1 and f_2 equal whenever $M[|f_1-f_2|^2]=0$, the set of bounded continuous functions forms a non-separable Hilbert space H in which $\bar{A}\varphi=M_t[\overline{K(t,s)}\varphi(t)]$. If K also satisfies $M_r[K(s,r)\overline{K(t,r)}]=M_r[\overline{K(r,s)}K(r,t)]$, the operator $A\varphi$ is normal and has a set of orthonormal proper functions $\{\varphi_n(t)\}$ which is complete in the range of A. The proper functions and values satisfy

$$\text{(1)}\qquad A\varphi_n=\lambda_n\varphi_n\quad\text{and}\quad\bar{A}\varphi_n=\bar{\lambda}_n\varphi_n$$

in the sense of equality in H.

The two basic definitions of the paper are now given. A family of generalized translations is a family T^s of one-to-one transformations which take $f(t)$ into $T_t^sf(t)=F_s(t)$ and which satisfy conditions A–F: (A) $T_t^0f(t)=T_0^tf(s)=f(t)$. (B) $|(T_t^s)^{\pm1}f(t)|\leqq M\sup|f(t)|$. (C) T_t^s is linear. (D) If $f(t)$ is real for all t, so is $T_t^sf(t)$. (E) $T_s^rT_t^sf(t)=T_t^sT_t^rf(t)$.

$$\text{(F)}\qquad M_r\{\overline{\bar{T}_r^sf(r)}\cdot\bar{T}_r^tf(r)\}=M_r\{\bar{T}_s^rf(s)\cdot\overline{\bar{T}_t^rf(t)}\}.$$

If T^s is also permutable with T^r and $\bar{T}^r$, the family is called commutative. Next, a continuous function $f(t)$ is defined to be almost periodic with respect to a family of translations T^s if the families of functions $T_t^sf(t)$ and $\bar{T}_s^tf(s)$ are both compact. It is now shown that, if $f(t)$ is a.p., $M[|f(t)|^2]>0$ unless $f(t)\equiv0$. Moreover the proper functions for the operator $A\varphi=M_t[\bar{T}_t^sf(t)\varphi(t)]$ satisfy equations (1) in the sense of equality for all values of the variable and not merely in the sense defined for H. Translations of proper functions are shown to be proper functions and their properties are studied. A function $e(t)$ is called a proper function of a commutative family of translations T^s if $T_t^se(t)=e(t)e(s)$. It is necessarily a.p. with respect to T^s, and satisfies $|e(t)|\leqq M$. The proper functions of T^s form an orthogonal set, and play the same rôle as complex exponentials in ordinary Fourier series. Thus if $f(t)$ is a.p. with respect to T^s, define

$$f(t)\sim\sum\frac{A_n}{\alpha_n}e_n(t),$$
$$A_n=\frac{1}{\alpha_n}M\{f(t)\overline{e_n(t)}\},\quad \alpha_n=(M\{|e_n(t)|^2\})^{\frac{1}{2}},$$

where $e_n(t)$ is the (necessarily denumerable) set of proper functions for which $A_n\neq0$. The author shows that Parseval's equation

$$M\{|f(t)|^2\}=\sum_{n=1}^{\infty}|A_n|^2$$

holds, as does also the uniform approximation theorem. Finally, properties A–F are illustrated with certain differential operators. *R. H. Cameron* (Cambridge, Mass.).

Cooper, J. L. B. The Fermi-Dirac functions. Philos. Mag. (7) **30**, 187–189 (1940). [MF 3029]

The bilateral Laplace transformation is employed to find the transform of the convolution integral defining the Fermi-Dirac functions of quantum statistics. Using this transform, known identities and asymptotic expansions for the functions are derived with ease. *J. L. Barnes.*

Meijer, C. S. Ueber eine Erweiterung der Laplace-Transformation. I. Nederl. Akad. Wetensch., Proc. **43**, 599–608 (1940). [MF 3102]

Meijer, C. S. Ueber eine Erweiterung der Laplace-Transformation. II. Nederl. Akad. Wetensch., Proc. **43**, 702–711 (1940). [MF 3107]

The author considers the reciprocal formulas

$$\text{(1)}\qquad f(s)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty K_\nu(s,t)(st)^{\frac{1}{2}}F(t)dt,$$

$$\text{(2)}\qquad F(t)=\frac{1}{i(2\pi)^{\frac{1}{2}}}\int_{\beta-i\infty}^{\beta+i\infty}I_\nu(ts)(ts)^{\frac{1}{2}}f(s)ds,$$

where K_ν and I_ν are the usual notations for Bessel functions of imaginary argument; the integral in (2) is a principal value. If $\nu=\pm\frac{1}{2}$, (1) reduces to the Laplace transform and (2) to the complex inversion formula for it; (1) and (2) are formally related to Hankel transforms as the Laplace transform and its inverse are related to Fourier sine and cosine transforms. Under suitable conditions on $F(t)$ (convergence of $\int e^{-\beta t}|F(t)|dt$, bounded variation), (1) implies (2) if $-\frac{1}{2}\leqq\nu\leqq\frac{1}{2}$; under suitable conditions on $f(s)$ (analyticity in a half-plane, order conditions), (2) implies (1) if $-\frac{1}{2}\leqq\Re(\nu)\leqq\frac{1}{2}$. Detailed statements of the conditions are too long to give here; they resemble familiar conditions for Laplace transforms. The author gives direct proofs of his theorems, using a considerable collection of lemmas on the functions I_ν and K_ν; he does not attempt to deduce his results from known conditions for the validity of Hankel's formula. Several reciprocal relations between special functions, previously given by the author, appear as special cases of (1) and (2). *R. P. Boas, Jr.* (Durham, N. C.).

Lambe, C. G. An infinite integral formula. Proc. Edinburgh Math. Soc. (2) 6, 75–77 (1939). [MF 1518]

The object of this note is to discuss the formula

$$(*)\qquad f(x+z)=\frac{1}{2\pi i}\int_{-\infty i}^{\infty i}\Gamma(-s)(-z)^s\{D^sf(x)\}ds,$$

the integral being supposed convergent for certain ranges of values of x and z. The contour is such that the poles of $\Gamma(-s)$ lie to its right and the other poles of the integrand to its left. $D^sf(x)$ denotes the sth differential coefficient with respect to x of $f(x)$ and this must be defined in an appropriate way for all values of s. The above formula is applied to the hypergeometric and Whittaker's confluent hypergeometric functions as well as to Legendre polynomials and Bessel functions. *A. Erdélyi* (Edinburgh).

Erdélyi, A. On Lambe's infinite integral formula. Proc. Edinburgh Math. Soc. (2) 6, 147–148 (1940). [MF 2845]

The author shows that under suitable conditions the relations

$$(1)\qquad \Gamma(-s)D^sf(x)=-\int_0^\infty f(x+z)(-z)^{-s-1}dz$$

and the formula (*) of the preceding review are equivalent; in fact, (1) and (*) are formally a pair of Mellin transforms, and the author obtains his conditions by applying standard theorems. It follows that in general no definition of the fractional derivative $D^sf(x)$ other than (1) is suitable for use in (*). The latter formula was not completely proved by Lambe. *R. P. Boas, Jr.* (Durham, N. C.).

Kharshiladze, F. I. On strong representation of a function by a singular integral. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. 6, 108–114 (1939). (Russian) [MF 3294]

The author says that $f(t)$ is weakly represented at $t=x$ by the singular integral with (non-negative) kernel $\Phi_n(t, x)$ if

$$(1)\qquad \lim_{n\to\infty}\int_a^b\{f(t)-f(x)\}\Phi_n(t, x)dt=0;$$

strongly represented, if

$$(2)\qquad \lim_{n\to\infty}\int_a^b|f(t)-f(x)|\Phi_n(t, x)dt=0.$$

He shows that if (1) is true almost everywhere for every integrable $f(t)$, then (2) is also true almost everywhere for every integrable $f(t)$. This is a generalization of the theorem that almost all the points where an integrable function is the derivative of its indefinite integral belong to its Lebesgue set; it is proved in the same way. The author obtains a simple sufficient condition for the strong representation of (t) at $t=x$ to imply that x belongs to the Lebesgue set of $f(t)$. This condition is that $\Phi_n(t, x)$ is an increasing function of t in $(x-1/n, x)$, and a decreasing function of t in $(x, x+1/n)$, and that $\Phi_n(x-1/n, x)\geqq n\alpha$, $\Phi_n(x+1/n, x)\geqq n\beta$ $(\alpha>0, \beta>0)$. This condition is satisfied by the Fejér and Weierstrass kernels; the Poisson kernel is also shown to have the property in question. Known representation theorems [for which the author could have given references earlier than the paper of Faddeeff which he cites], combined with the author's first theorem, thus show that an integrable function is strongly represented by the Fejér, Weierstrass or Poisson singular integrals at the points of its Lebesgue set, and at those points only. *R. P. Boas, Jr.*

Bourgin, D. G. Closure of products of functions. Bull. Amer. Math. Soc. 46, 807–815 and correction, 970 (1940). [MF 2927, 3458]

Let $s\sim(s_1, \cdots, s_m)$, $t\sim(t_1, \cdots, t_n)$ denote points in the euclidean spaces R_m, R_n. Let I_s, I_t denote generalized intervals in R_m, R_n and set $I_2=I_s\times I_t$, an interval in R_{m+n}. The author considers first the complex Hilbert space $L_2(I)$ of complex valued functions of summable square over I. The functions of the sequence $\{\varphi_\gamma(t)\psi_\mu(s)\}$, $\gamma, \mu=0, 1, \cdots$, are said to be closed (to form a fundamental set in the terminology of Banach) in $L_2(I_2)$ if and only if the linear combinations of functions in the sequence are dense in $L_2(I_2)$. The author then proves: (1) A necessary and sufficient condition that the sequence $\{\varphi_\gamma(t)\psi_\mu(s)\}$ be closed in $L_2(I_2)$ is that the sequences $\{\varphi_\gamma(t)\}$ and $\{\psi_\mu(s)\}$ be closed in the spaces $L_2(I_t)$ and $L_2(I_s)$, respectively. (2) If $\{F_\rho(s, t)\}$, $\rho=0, 1, \cdots$, is closed in $L_2(E_2)$, where E_2 is a Lebesgue measurable set in R_{m+n}, then the sequence is also closed in $L_2(E_s)$ except possibly for a t set of zero measure. The author next gives an abstract formulation in Banach space of these results and extends them. In the final theorem he obtains conditions under which the closure of $\{\varphi_\mu(t)\psi_\mu(s)\}$ implies the closure of $\{\psi_\mu(s)\}$ and shows that every finite subsequence of $\{\psi_\mu(s)\}$ is superfluous, that is, may be omitted without affecting the closure of the sequence.

G. B. Price (Lawrence, Kan.).

Widder, D. V. The Green's function for a differential system of infinite order. Proc. Nat. Acad. Sci. U. S. A. 26, 213–215 (1940). [MF 1606]

In a former paper [Trans. Amer. Math. Soc. 43, 7–60 (1938)] the author showed that the Stieltjes integral transform

$$f(x)=\int_0^\infty\frac{\phi(t)}{x+t}dt$$

can be inverted by a linear differential operator of infinite order which is formally equivalent to

$$\phi(x)=L[f(x)]=xD\prod_{n=-\infty}^{\infty}{}'\left(1-\frac{xD}{n}\right)f(x);$$

here D indicates differentiation with respect to x and the factor with $n=0$ is omitted. In this note the author considers the linear differential operator $L_k[f(x)]$ formed by the same product but for $n=-k, \cdots, k-2$. He constructs Green's function $G_k(x, t)$ for the equation $L_k[f(x)]=\phi(x)$ with the boundary conditions $\lim_{x\to+0}xf(x)=0$, $\lim_{x\to\infty}f(x)=0$ and states that

$$\lim_{k\to\infty}G_k(x, t)=\frac{1}{x+t}.$$

Hence this Stieltjes kernel may be regarded as the Green's function for a certain differential system of infinite order. The details will be given in a later paper.

E. Hellinger (Evanston, Ill.).

Geronimus, J. Sur les polynômes orthogonaux relatifs à une suite de nombres donnée et sur le théorème de W. Hahn. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 215–228 (1940). (Russian. French summary) [MF 2737]

Given a sequence of complex numbers $\{C_n\}$ such that all (Hankel) determinants $\Delta_n=|c_{i+k}|_{i,k=0}^{n-1}\neq 0$ $(n=1, 2, \cdots)$, we construct, in the known manner, a sequence of polynomials $\{P_n(z)\}$, of degree $n=0, 1, \cdots$, orthogonal in the following

general sense: $\sigma\{P_n(z)P_m(z)\}=0$ $(m\neq n)$ or $\Delta_{n+1}/\Delta_n\neq 0$ $(m=n)$. Here (Stieltjes) $\sigma\{\sum_{i=0}^{s} g_i z^i\}=\sum_{i=0}^{s} g_i c_i$ (and becomes the ordinary orthogonality if c_n is real and equal to $\int_{-\infty}^{\infty} x^n d\psi(x)$; $n=0, 1, \cdots$; $\psi(x)\uparrow$ and bounded). A simple study of the functional σ enables the author to establish necessary and sufficient conditions that each $P_n(z)$ shall be a linear combination of a finite number of polynomials belonging to another orthogonal sequence $\{P_n^*(z)\}$ constructed by means of $\{C_n^*\}$. This leads to necessary and sufficient conditions for the simultaneous orthogonality of $\{P_n(z)\}$ and $\{(1/n)P_n'(z)\}$ (a problem solved, under certain restrictive conditions, by W. Hahn, Krall, Webster), for here $P_n(z)$ turns out to be a linear combination of $P'_{n+1}(z)$, $P_n'(z)$, $P'_{n-1}(z)$. A further application is made to the study of the validity of the identity

$$\frac{1}{c_0}\sigma\left\{\frac{P_n(y)-P_n(z)}{y-z}\right\}=\frac{P_n'(y)}{n},\quad n=1, 2, 3, \cdots,$$

closely related to Tchebycheff formula of mechanical quadratures. Here the answer is: $\{P_n(y)\}$ are the trigonometric polynomials. [Reference should have been given here to R. P. Bailey ($c_n=\int_{-\infty}^{\infty} x^n d\psi(x)$), Duke Math. J. **2**, 287–303 (1936), and to M. Kravtchouk ($c_n=\int_{-\infty}^{\infty} x^n p(x)dx$, $p(x)\geqq 0$), Acad. Sci. Ukraine. J. Inst. Math. **2**, 87–92 (1934).]

J. Shohat (Philadelphia, Pa.).

Krall, H. L. On orthogonal polynomials satisfying a certain fourth order differential equation. Pennsylvania State College Studies, no. 6, 24 pp. (1940). [MF 2954]

In an earlier paper [Duke Math. J. **4**, 705–718 (1938)] the author has obtained in implicit form a necessary and sufficient condition that a certain type of linear homogeneous differential equation containing a parameter, a generalization of the equations of the second order satisfied by the "classical" orthogonal polynomials, have as solutions, for specified values of the parameter, a "Tchebycheff set" or "generalized orthogonal set" of polynomials. He showed there that the condition cannot be satisfied by an equation of odd order. The present paper is concerned with an explicit determination of the equations of the fourth order fulfilling the requirements. It is recognized in advance that certain equations of the fourth order satisfied by the Jacobi, Laguerre and Hermite polynomials, and obtained by iteration of their second-order equations, are included. It is found that all others can be reduced by linear transformation to one or another of four canonical forms.

D. Jackson (Minneapolis, Minn.).

Szegö, G. Remarks on a note of Mr. R. Wilson and on related subjects. Bull. Amer. Math. Soc. **46**, 852–858 (1940). [MF 2934]

The author first gives a rigorous and simple proof of the asymptotic relation

$$\lim_{n\to\infty}\max_{-1\leq x\leq 1}|p_n(x)|^{1/n}=1, \tag{1}$$

where $\{p_n(x)=k_n x^n+\cdots\}$, $n=0, 1, \cdots$, denote an orthonormal sequence of polynomials, with $\int_{-1}^{1} p_m(x)p_n(x)w(x)dx=\delta_{m,n}$. The weight-function $w(x)$ is subject to the condition

$$\int_{-1}^{1}\log w(x)(1-x^2)^{-\frac{1}{2}}dx \tag{2}$$

exists, which is less restrictive than that used by R. Wilson [ibid. **45**, 190–192 (1939)]. The author further shows that (2) is necessary (and not only sufficient) for the relation $2^{-n}k_n=O(1)$ [which implies (Shohat) the existence of $\lim_{n\to\infty} 2^{-n}k_n$]. This is an important supplement to an earlier result [Szegö, Shohat]. *J. Shohat* (Philadelphia, Pa.).

Broggi, Ugo. Contributo allo studio degli sviluppi in serie di polinomi di Laguerre. Ann. Mat. Pura Appl. (4) **19**, 141–150 (1940). [MF 3049]

If [$f(t)$ is real] $f(t)\sim\sum_0^\infty c_n L_n(t)$, and $\Gamma_n=c_0+c_1+\cdots+c_n$, then the convergence of $\sum\Gamma_n^2$ is necessary and sufficient in order that $f(t)$ shall be absolutely continuous, $f(t)=\int_0^t f'(u)du$, and $e^{-t/2}f(t)$ and $e^{-t/2}f'(t)\epsilon L_2(0, \infty)$. Further $\sum_0^\infty c_n L_n(t)$ converges to $f(t)$ for all $t\geqq 0$ and equals the sum of the uniformly convergent series $\sum_0^\infty \Gamma_n[L_n(t)-L_{n+1}(t)]$. Similar convergence conditions ensure the existence and quadratic integrability of higher order derivatives. [Cf. the reviewer's note in Proc. Nat. Acad. Sci. U. S. A. **12**, 348–352 (1926), where the necessity of the condition is proved as well as the convergence of the series $\sum_0^\infty c_n L_n(t)$ and its equivalence to the series $\sum_0^\infty \Gamma_n[L_n(t)-L_{n+1}(t)]$.] *E. Hille.*

Greenwood, Robert E., Jr. On Laguerre series. Proc. Nat. Acad. Sci. U. S. A. **26**, 466–471 (1940). [MF 2476]

The author discusses "Laguerre" series

$$f(s)=\sum a_n \exp(-\lambda_n s)L_\nu(2\lambda_n s),$$

where $L_\nu(z)$ is the Laguerre polynomial of νth degree (ν fixed). The λ_n form a strictly increasing sequence of positive numbers tending to infinity. It is attempted to show that the elements of the theory of Dirichlet series hold for this case; that is, (1) if the series converges at $s_0=\sigma_0+it_0$, $\sigma_0>0$, then the series is uniformly convergent in any angle $|\mathrm{am}\,(s-s_0)|\leqq\psi<\pi/2$; (2) if the series converges absolutely at $s_0(\sigma_0>0)$, then it converges absolutely in $\sigma\geqq\sigma_0$. The paper is carelessly written; lemma 2 is false as it is stated, and should not be used in the proof of theorem 3, but the results are not affected. The reviewer does not see why the author has restricted himself to a Laguerre polynomial.

F. Bohnenblust (Princeton, N. J.).

Shohat, J. Laguerre polynomials and the Laplace transform. Duke Math. J. 6, 615–626 (1940). [MF 2718]

Let $(1+z)A(z)=\int_0^\infty e^{-tu}f(u)du$, $t(1+z)=1$, $|z|<1$. The function $e^{-u/2}f(u)$ belongs to $L_2(0, \infty)$ if and only if $A(z)$ belongs to H_2. If the sequence $\{z_k\}$, $|z_k|<1$, satisfies certain simple conditions, the equations $A(z_k)=0$ imply $A(z)\equiv 0$. Combining these two facts, the author obtains results of Müntz and Szász on basis properties of the functions $\{x^{\mu_k}\}$ for the class $C(0, 1)$ or $L_2(0, 1)$. The reviewer refers to Pólya-Szegö, Aufgaben, vol. 2, p. 35, problem 198, where similar methods are used to the same purpose.

G. Szegö (Stanford University, Calif.).

Jackson, Dunham. Orthogonal polynomials with auxiliary conditions. Trans. Amer. Math. Soc. **48**, 72–81 (1940). [MF 2506]

The author deals with a set of polynomials orthogonal in a given interval with a given weight function such that the coefficients of each polynomial satisfy a finite number of preassigned linear conditions. The recursion formula and the Christoffel-Darboux identity are obtained. In the further course of the paper the expansion problem is discussed, especially in certain simple cases of linear conditions like $p_n'(1)=p_n'(-1)$ in which the polynomials $p_n(x)$ in question can be represented in terms of Legendre polynomials.

G. Szegö (Stanford University, Calif.).

Erdélyi, A. On some biorthogonal sets of functions. Quart. J. Math., Oxford Ser. **11**, 111–123 (1940). [MF 2609]

Using fractional integration by parts the author derives from a given biorthogonal set of functions new biorthogonal sets. Completeness of the original set also implies the same property for the new set. *O. Szász* (Cincinnati, Ohio).

Bell, E. T. Postulational bases for the umbral calculus. Amer. J. Math. **62**, 717–724 (1940). [MF 2872]

A rigorous foundation is given for the umbral calculus introduced by Blissard, supplementing the condensed treatment in the author's Algebraic Arithmetic. Possible confusion is avoided by the introduction of the new notations $\dot{+}$, $\{\ \}^N$, $(\ )^N$, $(\ )\cdot(\ )$, $A\cdot\alpha x$, and $x^{(A)}$. These special notations may be dropped when the distinctions emphasized by them have been understood. Examples are given in the new notation which incidentally show the usefulness of the calculus in manipulating formal power series.

O. Frink (State College, Pa.).

Hall, Newman A. A formal expansion theory for functions of one or more variables. Bull. Amer. Math. Soc. **46**, 824–832 (1940). [MF 2929]

The umbral calculus introduced by Blissard and developed by E. T. Bell is applied to the problem of expressing the expansion coefficients of a function f of several variables in terms of a given set of functions, in terms of the coefficients of the power series for f. The fundamental notion used is that of an associate set of functions $P_n(x)$ and $Q_n(y)$, the expansion coefficients of f in terms of the set $P_n(x)$ being expressed by means of the associate set $Q_n(y)$. The examples worked out include, among others, expansions in terms of Bessel functions, Jacobi polynomials, Gegenbauer functions, Hermite polynomials in two variables, products of Bessel functions and mixed products of Hermite and Gegenbauer polynomials. The question of convergence is not treated. *O. Frink* (State College, Pa.).

Integral Equations

Ingram, W. H. On the integral equations of continuous dynamical systems. Philos. Mag. (7) **30**, 16–38 (1940). [MF 2565]

The usual Schmidt reiterative process for locating the characteristic values and characteristic functions of an integral equation is presented as applicable in the case of complex characteristic values. The distribution of the roots of the approximating characteristic equations is simply assumed to be such as would give the desired result and the convergences are admitted to be "tentative." The formulas are applied to two examples. The author is unable to cope with theoretical difficulties due to his unawareness of the theory of "normalcy" and of the use of T^*T.

F. J. Murray (New York, N. Y.).

Doubrovsky, V. Sur les équations intégrales du type de Volterra correspondant aux espaces abstraits. Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 167–178 (1940). (Russian. French summary) [MF 2284]

Let A be an abstract space, $\mathfrak{M}$ a Borel field of subsets of A containing A and also all single points of A. Let $E\epsilon\mathfrak{M}$ be a fixed set, e a variable subset of E, $e\epsilon\mathfrak{M}$, $K(x, e)$ measurable in x for fixed e, and completely additive in e for a fixed x, $x\epsilon E$. The author considers generalized adjoint Volterra equations

$$u(x)=\lambda\int_{e(x)} K(x, dA_y)u(y)+f(x), \tag{1}$$

$$v(e)=\lambda\int_{e^*(e)} K(y, e)v(dA_y)+\phi(e), \tag{2}$$

where λ is constant, $f(x)$ measurable and bounded on E, $\phi(e)$ completely additive. The sets $e(x)$, $e^*(e)$ are assumed to satisfy the "property M": if $x_1\epsilon e(x)$ then $e(x_1)\subset e(x)$; if $e_1\subset e^*(e)$ then $e^*(e_1)\subset e^*(e)$. The author proves that under certain additional conditions equations (1), (2) possess a single solution $u(x)$, $v(e)$, respectively, which can be obtained by the classical method of successive approximation. As a typical case we quote only the following one: For every $x\epsilon E$ and every $\epsilon>0$ there exists a subdivision $e(x)=e_1+e_2+\cdots$ which has the property M and is such that $\bar{K}[e(x), e_i]<\epsilon$, $i=1, 2, \cdots$ (with the analogous condition where $e(x)$ is replaced by $e^*(e)$). Here $\bar{K}(e', e)$ denotes the "best majorant of $K(x, e)$, $x\epsilon e'$," that is, a completely additive function of e such that $\sup_{x\epsilon e'}|K(x, e)|\leqq\bar{K}(e', e)$ and such that whenever $\sup_{x\epsilon e'}|K(x, e)|\leqq K_1(e)$, then $\bar{K}(e', e)\leqq K_1(e)$.

J. D. Tamarkin (Providence, R. I.).

Doubrovsky, W. Sur certaines équations intégrales non-linéaires. Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 49–60 (1939). (Russian. French summary) [MF 2104]

It is proved that under suitable conditions on $f(x)$ the non-linear integral equation

$$u(x)=\lambda\int_E K(x, y, u(y))dy+f(x), \qquad x \text{ in } E,$$

where E is a bounded measurable set in the Euclidean space of one or more dimensions, has a solution (whatever the value of the parameter λ) provided that $K(x, y, u)$ satisfies appropriate conditions. Let $\phi(y, r, R)$ be the limit superior of $|K(x_1, y, u)-K(x_2, y, u)|$, when x_1, x_2 are points in E at distance from each other not greater than r and when $|u|\leqq R$. Some of the sufficient conditions are as follows: (1) $|K(x,y,u_1)-K(x,y,u_2)|\leqq C(x,y)|u_1-u_2|$, $0<\alpha<1$; (2) $\phi(y, r, R)\to 0$ (as $r\to 0$); (3) $\phi(y, r, R)$ is integrable with respect to y for r sufficiently small and for any $R>0$; (4) $K(x, y, 0)$ is integrable with respect to y for any x in E. An analogous study is made for the equation

$$u(x)=\lambda\int_E K(x, y)\varphi(y, u(y))dy+f(x).$$

In these developments an essential role is played by certain principles, due to Schauder and Cacciopoli, relating to fixed points in function spaces [Studia Math. **2** (1930); Atti Accad. Naz. Lincei. Rend. **11** (1930)].

W. J. Trjitzinsky (Urbana, Ill.).

Levi, Beppo. On the system $\int_{-\infty}^{+\infty}\phi(xy)dx=p(y)$; $\int_{-\infty}^{+\infty}\phi(xy)dy=q(x)$. Publ. Inst. Mat. Univ. Nac. Litoral **1**, no. 1, 8 pp. (1939). (Spanish) [MF 1655]

The author studies the pair of integral equations mentioned in the title, where $p(y)$ and $q(x)$ are given and $\varphi(x, y)$ is to be found. If a particular solution is known, the general solution is easily written down; hence the general solution is at once obtainable if the integrals $\int_{-\infty}^{\infty}p(y)dy$ and

$\int_{-\infty}^{\infty} q(x)dx$ exist and are equal. The author obtains explicitly a (non-unique) solution when $p(y)=y^n$, $q(x)=0$, and hence when $p(y)$ and $q(x)$ are any polynomials. The solution is easily obtained once coefficients a_m have been determined so that

$$\varphi(x, y)=\sum_{m=0}^{n} a_m x^m y^{n-m} e^{-(x+y)^2}$$

is a solution of the system with $p(y)=y^n$, $q(x)=x^n$.

R. P. Boas, Jr. (Durham, N. C.).

Fortet, Robert. Résolution d'un système d'équations de M. Schrödinger. J. Math. Pures Appl. **19**, 83–105 (1940). [MF 1893]

To find positive solutions of the pair of integral equations

$$\text{(S)}\qquad \begin{aligned} \varphi(x)\int_c^d g(x, y)\psi(y)dy&=\omega_1(x),\\ \psi(y)\int_a^b g(x, y)\varphi(x)dx&=\omega_2(y), \end{aligned}$$

where $\omega_1(x)\geqq 0$, $\omega_2(x)\geqq 0$ and continuous,

$$\int_a^b \omega_1(x)dx=\int_c^d \omega_2(y)dy=1,$$

$g(x, y)\geqq 0$ continuous, has an upper bound and does not vanish for any fixed value of x or y except in points of a set of measure zero. Theorem I. If

$$\int_c^d \omega_2(y)\left\{\int_a^b g(x, y)\omega_1(x)dx\right\}^{-1} dy$$

is finite, the system (S) has a positive solution, where $\varphi(x)$ is continuous and vanishes only at the zeros of $\omega_1(x)$, while $\psi(y)$ is measurable and its zeros, apart from a set of measure zero, are identical with those of $\omega_2(y)$. Theorem II. Suppose that $g(x, y)$ has the following properties: Let J be any closed interval in (a, b). Let $g(x, y)\geqq 0$ and continuous in J, and let the hypothesis that the integral

$$\int_c^d g(x, y)f(y)dy$$

be convergent almost everywhere in any open interval contained in J (where $f(y)$ is any nonnegative continuous function) imply that this integral be uniformly convergent in J. Then (S) has a solution such that $\varphi(x)=0$, where $\omega_1(x)$ and $\varphi(x)>0$ and continuous for all other values of x, $\psi(x)$ measurable and does not vanish, except on a set of measure zero, and where $\omega_2(y)=0$. Theorem III. The system (S) has not more than one positive and measurable solution. Bernstein's result for the particular case

$$g(x, y)=\frac{1}{\sigma\pi^{\frac{1}{2}}} e^{-(x-y)^2/\sigma^2}$$

[Verh. Intern. Math. Kongr. Zürich, 1932, vol. **1**, p. 288] is contained among the present ones. *A. Erdélyi.*

Michlin, S. Sur une certaine classe d'équations intégrales singulières. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 315–317 (1939). [MF 2045]

Let

$$p(\phi)=\frac{1}{\pi}\int_C \frac{\phi(y)}{y-x}dy,$$

the integral being taken in the Cauchy principal value sense, and where C is a closed curve suitably restricted. Also let $f(x, \phi)$ be a sum of a known function in L_2 and a completely continuous linear operator from L_2 to L_2. The author considers the functional equation

$$(*)\qquad \phi(x)-\lambda b(x)p(\phi)=f(x, \phi),$$

where $b(x)$ is a given complex valued function. If $D^{(\infty)}$ is the set of points in the region defined by $\lambda=\pm 1/b(x)$, containing neither $\lambda=0$ nor $\lambda=\infty$, then necessary and sufficient conditions are obtained in order that the equation (*) be equivalent to an ordinary Fredholm equation. [Cf. a paper by the author, C. R. (Doklady) Acad. Sci. URSS (N.S.) **15**, 429–432 (1937).] *I. A. Barnett* (Cincinnati, Ohio).

Smithies, F. Singular integral equations. Proc. London Math. Soc. (2) **46**, 409–466 (1940). [MF 2824]

Three classes of the integral equation

$$f(x)=\int_a^b K(x-y)f(y)dy$$

are discussed: (i) the Volterra case, where the limits of the integral are $(-\infty, x)$; (ii) the Fredholm case with $(-\infty, \infty)$; (iii) the Wiener-Hopf case with $(0, \infty)$. By a refinement of the method introduced by Wiener and Hopf [S.-B. Preuss. Akad. Wiss. **1931**, 696–706] the author succeeds in showing, under mild restrictions, the exhaustiveness of the known solutions. Put

$$\kappa(s)=\int_{-\infty}^{+\infty} K(x)e^{-sx}dx,$$

where in the Volterra case $K(x)\equiv 0$ for $x<0$. Throughout the paper $K(x)$ is subjected to a condition which, in essence, requires that the characteristic equation $\kappa(s)=1$ has no root with $\Re(s)=0$ and only finitely many roots $s_1, \cdots, s_n$ with $\Re(s_\nu)>0$. Each of the three classes is divided into two types. In the Volterra equations of the first type it is assumed that $K(x)=K_1(x)+K_2(x)$, where $K_1(x)\varepsilon V(0, \infty)$ and $K_2(x)\varepsilon L(0, \infty)$. It is shown that, if a solution $f(x)\varepsilon L(-\infty, X)$ for every X, then $f(x)=\sum_1^n P_\nu(x)e^{s_\nu x}$, where $P_\nu(x)$ is a polynomial of degree less than the multiplicity of the root s_ν. The converse is, of course, also true. For the second type the assumptions are more complicated; roughly speaking they require that $K(x)$ behaves, for $x\to\infty$, like x^{-k} for some $k>0$. Then a similar theorem holds for all solutions $f(x)$ such that $\{1+|x|^k\}^{-1}f(x)\varepsilon L(-\infty, X)$.

In the Fredholm and Wiener-Hopf equations of the first type it is assumed that $K(x)=K_1(x)+K_2(x)$, where $K_1(x)e^{|x|}\varepsilon V(-\infty, \infty)$ and $K_2(x)e^{|x|}\varepsilon L(-\infty, \infty)$. Again it is shown that there is only a finite number of linearly independent solutions such that $f(x)e^{-|x|}\varepsilon L(-\infty, \infty)$ (in the Wiener-Hopf case one puts, of course, $f(x)=0$ for $x<0$). In the equations of the second type $K(x)$ behaves, roughly speaking, like $|x|^{-k}e^{-|x|}$. Among the special equations treated we mention those of Picard, Lalesco and Milne (for the radiative equilibrium of stars). *W. Feller.*

Trjitzinsky, W. J. Some problems in the theory of singular integral equations. Ann. of Math. (2) **41**, 584–619 (1940). [MF 2555]

While Trjitzinsky in his extension of T. Carleman's theory of singular integral equations [Trans. Amer. Math. Soc. **46**, 202–279 (1939); these Rev. **1**, 17] investigated the solution of the nonhomogeneous equation

$$(1)\qquad \phi(x)-\lambda\int_0^1 K(x, y)\phi(y)dy=f(x)$$

only for non-real values of the parameter λ as did Carleman, he now succeeds in finding results for real values λ; this is important not only for the theory but also for the applications. He considers three different types of real symmetric kernels $K(x, y)$: type (A) corresponds essentially to the class K_1 of the preceding paper which was first treated by Carleman; (B) contains all kernels for which $\int_0^1 K(x, y)^2 dy$ exists for almost all $0 \leqq x \leqq 1$; (C) embraces all kernels for which an "associated linear operator," as defined in the preceding paper, exists. Approximating $K(x, y)$ by a sequence of bounded kernels $K_n(x, y)$ with the characteristic values $\lambda_{n\nu}$, he makes the assumption that there exists a closed real interval d, which contains neither values $\lambda_{n\nu}$ nor their limiting points; he considers any set S_d of points λ consisting of such an interval d and of all non-real values. Investigating the solutions of the nonhomogeneous equations with the kernels $K_n(x, y)$ and the right member $f(x)$, and choosing suitable subsequences, he proves: For kernels of type (A), (B), there exists a solution of (1) for almost all $0 \leqq x \leqq 1$, and this solution is analytic in λ for λ in S_d. For the type (C) the analogous result holds if (1) is transformed in a suitable manner by the associated linear operator. In all cases a spectrum expression for the solution is given. Furthermore, using some important properties of the linear operators arising from the integral equation, he discusses questions of uniqueness and he finds interesting results corresponding to Carleman's results for non-real λ. As an instance the following may be quoted: If (1) with a kernel (A) or (B) has, for a fixed λ in S_d, only one solution for which $\phi(x)$, $\phi(x)^2$ are summable, the same will be true for all non-real values of λ. The results for kernels of type (B) can be extended to such sets of real values, instead of the interval d, which contain no characteristic value $\lambda_{n\nu}$ but may contain their limiting points. Finally, some analogous results for integral equations of the first kind are given; here the results for the type (C) cannot be found by means of the classical methods of Picard and Lauricella.

E. D. Hellinger (Evanston, Ill.).

Winton, Lowell S. A compatible integro-differential system. Duke Math. J. 6, 562–578 (1940). [MF 2715]

The author is concerned with the derivation of a generalized Green's matrix $\|H_j^i(x, y : s)\|$ for an integro-differential system of the form $L^i[u(x:s)] = f^i(x:s)$, $U^i[u(x:s)] = X^i(s)$, where

$$L^i \equiv \frac{\partial u^i(x:s)}{\partial x} + \Phi_j^i(x:s)u^j(x:s) + \int_\alpha^\beta K_j^i(x:s,t)u^j(x:t)dt,$$

$$U^i \equiv \alpha_j^i(s)u^j(a:s) + \beta_j^i(s)u^j(b:s) + \int_\alpha^\beta [A_j^i(s,r)u^j(a:r) + B_j^i(s,r)u^j(b:r)]dr,$$

and under the hypothesis that the homogeneous system $L^i[u(x:s)] = 0$, $U^i[u(x:s)] = 0$ is compatible.

W. T. Reid (Chicago, Ill.).

Feller, Willy. On the integro-differential equations of purely discontinuous Markoff processes. Trans. Amer. Math. Soc. 48, 488–515 (1940). [MF 3169]

The author makes a searching analysis of the conditional probability functions determining a Markoff process proceeding by jumps. Let $P(\tau, x; t, A)$ be the probability that a system in state x at time τ will be in a state of the set A at time $t > \tau$. It is supposed that P is completely additive in A, with $0 \leqq P \leqq 1$, $P(\tau, x; t, E) \equiv 1$ (where E is the whole of x-space). The critical regularity conditions imposed on P are (besides the details of measurability, etc.) (1) if $\tau \uparrow t$ or $t \downarrow \tau$, $P \to \delta(x, A)$ ($=1$ if $x \epsilon A$, $=0$ otherwise); (2) P satisfies the Chapman-Kolmogoroff equation [cf. Kolmogoroff, Math. Ann. 104, 415–458 (1931)]; (3) $P(\tau, x; t, A) = [1 - p(t,x)(t-\tau)]\delta(x, A) + p(t,x)\Pi(t, x, A)(t-\tau) + o(t-\tau)$, where (3a) $p \geqq 0$, $0 \leqq \Pi \leqq 1$, Π is completely additive in A, $\Pi(t, x, x) \equiv 0$, $\Pi(t, x, E) \equiv 1$. Condition (3) is imposed so that the system will change by jumps: neglecting $o(\Delta)$ terms, $1 - p(t,x)\Delta$ is the probability that no change (from x) will occur in the interval $(t, t+\Delta)$, and $\Pi(t, x, A)$ is the probability that, if a change occurs, the state will go from x into A. The following facts are shown, generalizing and completing earlier results by the author [Math. Ann. 113, 113–160 (1936)] and by Dubrowski [C. R. (Doklady) Acad. Sci. URSS (N.S.) 19, 439–446 (1938)]. Related work with more stringent hypotheses has been done by Pospíšil [Časopis Pěst. Mat. Fys. 65, 64–76 (1936)] and Doeblin [Skand. Aktuarietidskr. 22, 211–222 (1939); these Rev. 1, 247]. The essential significance of the results lies in the precise relations found between the character of the distribution functions P and the stochastic process in the large, with the relations in the small determined by p, Π. It is proved that P satisfies the two adjoint integro-differential equations

$$\frac{\partial P(\tau, x; t, A)}{\partial t} = -\int_{A_y} p(t, y)P(\tau, x; t, dE_y) + \int_E p(t, y)\Pi(t, y, A)P(\tau, x; t, dE_y),$$

$$\frac{\partial P(\tau, x; t, A)}{\partial \tau} = p(\tau, x)\left\{P(\tau, x; t, A) - \int_E P(\tau, y; t, A)\Pi(\tau, x, dE_y)\right\}. \tag{4}$$

An examination of these equations shows that, under the present conditions, each of them determines uniquely one and the same function $P(\tau, x; t, A)$; for this function two representations by means of infinite series are deduced. It follows that if functions Π, p are given, satisfying conditions (3a) and elementary regularity conditions, there is a corresponding P, satisfying (1), (2), (3) and the other conditions outlined above, except that $P(\tau, x; t, E)$ may be <1. This case will arise if the transition times of the system are not isolated. Conditions are given that this case should not occur, and necessary and sufficient conditions that this case should not occur are found in the temporally homogeneous case (when p, Π do not depend on t).

J. L. Doob (Urbana, Ill.).

Smirnoff, N. Sur l'application des séries de Fourier à la résolution des équations intégrales et intégrodifférentielles. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 1939, 413–428 (1939). (Russian. French summary) [MF 2088]

The author proves that, if $F_1(x)$, $F_2(x)$ belong to L^2 and have period 1, then their convolution

$$F_1 * F_2 = \int_0^1 F_1(x-u)F_2(u)du$$

has as its Fourier coefficients the product of the Fourier coefficients of F_1 and F_2. This fact is not, as the author seems to believe, new [see, for example, N. Wiener, The

Fourier Integral, 1933, p. 45]. He uses it to reduce to the solution of algebraic equations, involving the Fourier coefficients of the unknown function $u(x)$, the solution of such integral equations as

$$u(x)=f(x)+\lambda\int_0^1 K(x-y)u(y)dy,$$

$$f(x)+\sum_{k=0}^{m} A_k u^{(k)}(x)=\sum_{l=0}^{n}\int_0^1 K_l(x-y)u^{(l)}(y)dy,$$

$$f(x)=\sum_{k=1}^{\infty}\int_0^1\cdots\int_0^1 K_k(x-x_1-\cdots-x_k)u(x_1)\cdots u(x_k)dx_1\cdots dx_k.$$

R. P. Boas, Jr. (Durham, N. C.).

Bernstein, S. Sur une classe d'équations fonctionnelles aux dérivées partielles. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 17–26 (1940). (Russian. French summary) [MF 1979]

The author considers the boundary value problem

$$\frac{\partial^2 u}{\partial t^2}=\varphi\left(\int_0^l\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$u(0,t)=u(l,t)=0$, $u(x,0)=F(x)$, $\partial u(x,0)/\partial t=F_1(x)$ (l a constant; $\varphi(y)\geqq\alpha>0$ continuous, $\varphi^{(1)}(y)$ finite for $y\geqq 0$), where $F(x)$, $F_1(x)$ are given functions in the form of Fourier sine series. It is proved that there is a solution $u=\sum A_k(t)\sin kx$ (if $l=\pi$), where the $A_k(t)$ satisfy an infinite system of differential equations of second order, provided that the Fourier coefficients of F, F_1 approach zero sufficiently fast; moreover, t is to be sufficiently small. The same method can be applied in the case when in the second member of (1) $\partial u/\partial x$ is replaced by u or by $\partial^p u/\partial x^p$ ($p>1$). The corresponding two problems for the equations

$$\frac{\partial u}{\partial t}=\varphi\left(\int_0^\pi\left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2},\quad \frac{\partial^2 u}{\partial t^2}=\varphi\left(\int_0^l\left(\frac{\partial u}{\partial x}\right)^2 dx\right)u$$

are also considered. *W. J. Trjitzinsky* (Urbana, Ill.).

Functional Analysis, Ergodic Theory

Day, Mahlon M. The spaces L^p with $0<p<1$. Bull. Amer. Math. Soc. 46, 816–823 (1940). [MF 2928]

The author considers an abstract set E containing an additive family of subsets and an additive non-negative set-function defined on this family. An integral can then be defined, and it has many of the properties of the Lebesgue integral. [Saks, Theory of the Integral, Chapter I.] The space $L^p(0<p<1)$ is the class of all measurable functions defined on E and having their pth power integrable. The author indicates how the usual topology of this space can be given an equivalent Fréchet metric. The principal result of the paper is that there is a non-zero linear and continuous functional on this space if and only if the set E contains no subsets which are singular in the sense of Saks [Trans. Amer. Math. Soc. 35, 965–970 (1933)]. This result shows that when E is a measurable subset of a Euclidean space no interesting theory of linear continuous functionals can be obtained. A second result of the author is to obtain a representation theorem for linear continuous functionals in the case when there are singular sets. This representation is by an infinite series. *H. H. Goldstine.*

Bohnenblust, F. An axiomatic characterization of L_p-spaces. Duke Math. J. 6, 627–640 (1940). [MF 2719]

L_p is partially ordered by the definition $f_1\leqq f_2$ if $f_1(t)\leqq f_2(t)$ for almost every t; f_1 and f_2 are said to be orthogonal if essentially $f_1(t)\cdot f_2(t)=0$. (This notion can be defined in terms of the l.u.b. and gr.l.b. of two functions and hence in terms of the ordering.) The ordering determines and is determined by the Boolean algebra associated with the measurable sets. Similar notions apply in the spaces $l_{p,n}$, l_p, $L_p\oplus_p l_{p,n}$ and $L_p\oplus_p l_p$. These spaces also have the property (P): "If f_1 and f_2 are orthogonal, $\|f_1+f_2\|$ depends only on $\|f_1\|$ and $\|f_2\|$." If $F(\eta,\xi)$ is a function of two positive real variables, homogeneous of degree one, monotonic in each variable for every value of the other, symmetric, with $F(0,1)=1$ and such that $F(\xi,F(\eta,\zeta))=F(F(\xi,\eta),\zeta)$, then $F(\xi,\eta)=(\xi^p+\eta^p)^{1/p}$ for a p such that $0<p\leqq\infty$. (For $p=\infty$, $F(\xi,\eta)=\max(\xi,\eta)$.) When a partially ordered space has property (P) and f_1 and f_2 are orthogonal, then $\|f_1+f_2\|$ is such a function of $\|f_1\|$ and $\|f_2\|$. The resulting p and the Boolean algebra characterize such a space. If $p<\infty$ and the space is separable it is one of the types mentioned. A separable space with $p=\infty$ must be finite dimensional or c_0 the space of sequences which converge to zero. The non-separable cases are also discussed. *F. J. Murray.*

Komatuzaki, Hitosi. Sur les projections dans certains espaces du type (B). Proc. Imp. Acad. Tokyo 16, 274–279 (1940). [MF 2945]

It is shown that there exist manifolds in (C), (c), (M), (m), (C^p), $p=1,2,\cdots$, and (c_0) which do not possess projections. The corresponding result for L which is due to Banach is used and certain methods of the reviewer. [Cf. Trans. Amer. Math. Soc. 41, 138–152 (1937) and Studia Math. 6, 199–211 (1936).] *F. J. Murray.*

Šmulian, V. Über lineare topologische Räume. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 425–448 (1940). (German. Russian summary) [MF 2801]

If E is a locally convex linear topological space and Γ a linear and total set of functionals on E, one may, as in a Banach space, define a Γ-weak topology in E. For this rather general situation theorems are proved which, when specialized by the proper choice of E and Γ, give results concerning Banach spaces previously proved by the author and others. In this way the author obtains several necessary and sufficient conditions for reflexiveness (regularity) and for regular closure and regular convexity [Krein and Šmulian, Ann. of Math. (2) 41, 556–583 (1940); these Rev. 1, 335]. There are also several theorems in which it is shown that certain properties of a set in a space E can be reformulated by considering the set in E with one of its weak topologies; e.g., the regularly convex sets of E^* (norm topology) coincide with the convex and closed sets of E^* with its E-weak topology. In a final section a "weak" topology is considered which, in a Banach space, is stronger than the norm topology. *J. V. Wehausen.*

Šmulian, V. Sur la dérivabilité de la norme dans l'espace de Banach. C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 643–648 (1940). [MF 3226]

For an arbitrary set Q let $E(Q)$ be the linear normed space of bounded real functions on Q, where $\|x\|=\sup|x(q)|$. The necessity of the following condition is proved; sufficiency was proved previously [the same C. R. 24, 648–652 (1939); cf. these Rev. 1, 242]. Let $x_0\varepsilon E(Q)$, $\|x_0\|=1$. Then,

in order that the strong differential of the norm $\|x\|$ in $E(Q)$ exist at the point x_0, it is necessary and sufficient that the following condition be satisfied. For every extremal sequence $\{q_n\} \subset Q$ of the function $x_0(q)$ (that is, $\|x_0\| = |\lim x_0(q_n)|$) and for every $x(q) \epsilon E(Q)$ with $\|x\| \leqq 1$, the sequence $\{x(q_n)x_0(q_n)\}$ converges uniformly in the unit sphere of $E(Q)$ toward a limit which is independent of the particular choice of extremal sequence. For any normed linear space the theorem is applied to E or $\bar{E}$, respectively, by taking Q as the surface of the unit sphere in $\bar{E}$ or E, respectively. It is also proved that a Banach space E is regular if the strong differential exists at every point of $\bar{E}$ and $\bar{\bar{E}}$.

Additional notions of uniform weak and uniform strong differentiability are introduced and investigated. It is shown that if its norm is everywhere uniformly strongly differentiable a Banach space is regular. *J. V. Wehausen.*

Michal, A. D. Differentials of functions with arguments and values in topological abelian groups. Proc. Nat. Acad. Sci. U. S. A. **26**, 356–359 (1940). [MF 2145]

For functions whose domain and range are topological groups various definitions for a differential exist. Two of these are due to Michal, while in linear spaces there is the Fréchet differential. The present paper discusses, without proofs, the equivalence of these definitions, their topological invariance, the differential of a function of a function and higher order differentials. *F. J. Murray.*

Jeffery, R. L. Integration in abstract space. Duke Math. J. **6**, 706–718 (1940). [MF 2723]

Assuming that X is a linear normed complete space and E is a bounded Lebesgue measurable subset of a Euclidean space, the bounded function f on E to X is said to be integrable if there exists a sequence of partitions $\Pi^{(n)}(E)$ into measurable subsets $e_1^{(n)}, \cdots, e_i^{(n)}, \cdots$ such that $\lim_n me_i^{(n)} = 0$ for each i, and $\lim_n \sum_i f(\xi_i^{(n)})me_i^{(n)}$ exists, $\xi_i^{(n)}$ ranging over $e_i^{(n)}$. If two sequences of partitions each provide a limit, the limits are equal. If f is unbounded on E, it is integrable if f is integrable on every bounded subset G of E and $\lim_{mG\to mE} T(f, G)$ exists, $T(f, G)$ the integral on G. A bounded function is said to be restricted if there exists an R such that for every $\epsilon > 0$ there exists a set of disjoint sets e_i such that $me_i < \epsilon$, $\sum_i me_i = mE$ and $\|\sum_i (f(\xi_i) - f(\xi_i'))\| < R$, ξ_i, ξ_i' any points of e_i; f is measurable if it is a limit of a sequence of restricted functions. A measurable function for which $\int_E \|f\|$ exists, is integrable. Functions assuming at most a finite number of values are restricted. As a consequence functions integrable by the definition of Bochner [Fund. Math. **20**, 262–276 (1933)] are integrable according to the present definition. It is proved that the definition of integrability due to G. Birkhoff [Trans. Amer. Math. Soc. **38**, 357–378 (1935)] is equivalent to that of this paper.
T. H. Hildebrandt (Ann Arbor, Mich.).

Phillips, R. S. Integration in a convex linear topological space. Trans. Amer. Math. Soc. **47**, 114–145 (1940). [MF 1073]

Let Σ be an additive family of subsets σ in an abstract set S, with $\alpha(\sigma)$ non-negative, bounded, and completely additive on Σ. Let X be a convex linear topological space of the Kolmogoroff type and let $\mathfrak{B} = [U]$ be a complete system of neighborhoods of zero. Denoting finite subsets of the positive integers by π, the author calls a formal series $\sum_1^\infty X_i$ of subsets of X unconditionally summable to $x \epsilon X$ with respect to a fixed $U \epsilon \mathfrak{B}$ if there is a π_U such that $\pi \geqq \pi_U$ implies $\sum_\pi x_i - x \epsilon U$ for all choices of $x_i \epsilon X_i$. Using this to replace unconditional convergence in G. Birkhoff's integral for Banach space valued functions [Trans. Amer. Math. Soc. **38**, 357–378 (1935)], the author defines as follows a Lebesgue type integral for functions $x(\sigma)$ defined on Σ to the space of subsets of X with $x(\sigma_1) \supset x(\sigma_2)$ whenever $\sigma_1 \supset \sigma_2 : x(\sigma)$ is $\mathfrak{B}$-integrable if for each $\sigma \epsilon \Sigma$ there exists $J(x, \sigma) \epsilon X$ such that given $U \epsilon \mathfrak{B}$ there are disjoint sets $\sigma_i \epsilon \Sigma$ with the properties that $\sum_1^\infty \sigma_i = S$ and $\sum_1^\infty x(\sigma \cdot \sigma_i)\alpha(\sigma \cdot \sigma_i)$ is unconditionally summable to $J(x, \sigma)$ with respect to U. The integral of $x(\sigma)$ is $J(x, \sigma)$. [G. Birkhoff, Ann. of Math. (2) **38**, 39–56 (1937), in particular pp. 50–52.] From two preliminary basic theorems on uniformity it follows that (1) $J(x, \sigma)$ is single-valued, completely additive, bounded, and absolutely continuous in the sense that $\alpha(\sigma) < \delta_U$ implies $J(x, \sigma) \epsilon U$, (2) when T is distributive and continuous on X to Y the transform $T[x(\sigma)]$ is integrable with respect to the neighborhood system in Y and $J(T[x], \sigma) = T[J(x, \sigma)]$, (3) the most general convergence theorem for Lebesgue integrals remains true in this setting.

When X is a Banach space it is easily shown that by varying $\mathfrak{B}$ the present integral, due to its ingenious definition, becomes any one of several earlier integrals—G. Birkhoff's when $\mathfrak{B}$ is the norm topology, that discussed by the reviewer [Trans. Amer. Math. Soc. **44**, 277–304 (1938)] when $\mathfrak{B}$ is the weak neighborhood topology defined by all finite sets of linear functionals over X, and those of Gelfand [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff **13**, 35–40 (1936)] and Dunford [Trans. Amer. Math. Soc. **44**, 305–356 (1938)] when $\mathfrak{B}$ is defined by all finite subsets of certain subclasses of linear functionals. In differentiation previous results are included in the statement that a single-valued additive absolutely continuous function $x(\sigma)$ of measurable sets in $(0, 1)$ is (with respect to $\mathfrak{B}$) pseudo-differentiable and of bounded variation if and only if it is a $\mathfrak{B}'$-integral, the latter being a stronger version of the $\mathfrak{B}$-integral and analogous to Bochner's integral for Banach spaces [Fund. Math. **20**, 262–276 (1933)]. It is also shown that, if $x_0 \epsilon X$ and $\phi(x, s)$ on $X \times (0, 1)$ to X is continuous in x, measurable in s, and has its set of values compact, there is an $x(s)$ such that $x(s) = x_0 + \int_0^s \phi(x(t), t)dt$, the integral being the $\mathfrak{B}'$. The paper concludes with an interesting list of examples, including one of a G. Birkhoff integral in Hilbert space that a.e. fails to be weakly differentiable. *B. J. Pettis.*

Wong, Y. K. On the converse of the transitivity of modularity. Bull. Amer. Math. Soc. **46**, 352–355 (1940). [MF 1847]

In a basis involving the number system of the general set $\mathfrak{P}$ and two positive hermitian matrices ϵ and ϵ_0, on $\mathfrak{P}\mathfrak{P}$ to $\mathfrak{A}$, the author proves the equivalence of three properties: (1) every vector μ_0 modular (or limited) as to ϵ_0 is modular as to ϵ, (2) ϵ_0 is modular as to $\epsilon\epsilon_0$, and (3) ϵ_0 is modular as to $\epsilon\epsilon$, the result: (2) implies (1) being due to E. H. Moore [General Analysis, vol. 2, p. 84]. A similar theorem holds with respect to a matrix κ^{12} on $\mathfrak{P}^1\mathfrak{P}^2$ to $\mathfrak{A}$ relative to hermitian matrices $\epsilon^1, \epsilon^2, \epsilon_0^1, \epsilon_0^2$, the statements being in this case (1) every matrix κ^{12} modular as to $\epsilon_0^1, \epsilon_0^2$ is modular as to ϵ^1, ϵ^2, (2) every matrix κ^{12} modular as to $\epsilon_0^1, \epsilon_0^2$ is of type $\mathfrak{M}^1\overline{\mathfrak{M}}^2$, (3) ϵ_0^1 is modular as to $\epsilon^1\epsilon^1$ and ϵ_0^2 is modular as to $\epsilon^2\epsilon^2$. There follows a generalization of the Hellinger-Toeplitz theorem in the form: a matrix κ^{12} is modular as to $\epsilon^1\epsilon^2$ if and only if κ^{12} is by rows modular as to ϵ^2 and $J^2\kappa^{12}\mu^2$ is modular as to ϵ^1 for every μ^2, where J^2 is the Moore integration process relative to ϵ^2. *T. H. Hildebrandt.*

Kakutani, Shizuo. Simultaneous extension of continuous functions considered as a positive linear operation. Jap. J. Math. 17, 1–4 (1940). [MF 3109]

Let R be a metric space, and let E be a closed subset of R. Let $C(R)$ and $C(E)$ be the Banach spaces of real valued continuous bounded functions over R and E, respectively. Any $f\epsilon C(R)$, considered as a function over E, determines an element $g\epsilon C(E)$, $g=A(f)$. The operation A is positive linear and of norm 1. The study of the inverses to A is the problem of extending to R bounded continuous functions over E. It is known that any bounded continuous function over E can be extended continuously to R. Even more, an inverse operation of A can be given which is positive and isometric, although linearity may not be satisfied. The present author verifies by a simple construction that, under the added assumption that E be locally separable, an inverse of A exists which is also linear. It is not known whether this added assumption is essential for the validity of the theorem.

F. Bohnenblust (Princeton, N. J.).

Rutman, M. A. Sur les opérateurs totalement continus linéaires laissant invariant un certain cône. Rec. Math. [Mat. Sbornik] N.S. 8 (50), 77–96 (1940). (French. Russian summary) [MF 3191]

The paper generalizes to a Banach space B the following theorem on square matrices due to Perron [Math. Ann. 64, 1–76 (1907)] and Frobenius [S.-B. Preuss. Akad. Wiss. 1907, 1908, 1912]: If for some integer n the nth power of the positive matrix A is positive definite, then the characteristic number of largest absolute value is positive and has multiplicity 1, and the corresponding characteristic vector is positive. To this end there is defined the notion of a cone K, namely, a closed set of elements, positive linear in the sense that, for x_1x_2 of K and $\lambda_1, \lambda_2\geqq 0$, $\lambda_1x_1+\lambda_2x_2$ belongs to K, and such that there exists a linear functional f so that $f(x)>0$ for all x of K except zero. A positive operation is defined to satisfy the conditions that A is on K to K and, for some x_0 of K and $c>0$, $Ax_0-cx_0\epsilon K$. Such an operation, if totally continuous, possesses in K a characteristic vector. If in addition K has the property that any element of E is the difference of two elements of K, then there exists among the characteristic numbers of largest absolute value a positive one with characteristic vector in K. If the cone K possesses interior points and if A on K to K is also totally positive in the sense that there exists an integer n so that A^n transforms points of the frontier of K into interior points, then A possesses in the cone a single characteristic vector whose characteristic value λ_0 is the characteristic number of largest absolute value of A and the point $1/\lambda_0$ is a simple pole of the resolvent of A.

T. H. Hildebrandt (Ann Arbor, Mich.).

Kitagawa, Tosio. The characterisations of the fundamental linear operations by means of the operational equations. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. 1, 1–28 (1940). [MF 2668]

The author considers operations in the space $L^{(2)}$ of functions which are square summable on every finite interval. The operations considered take functions of a fixed finite interval into those on another finite interval and for each interval the operation A satisfies a condition $\|Af\|\leqq C\|f\|$ or $\|Af\|\leqq C_1\|f\|+C_2\|f'\|$, where the norms are the usual L_2 norms for a function defined on an interval. The author stresses the use of $L^{(2)}$ and claims that this approach is more direct than the use of unbounded closed operators in L_2 for the interval $(-\infty, \infty)$. Operational equations involving multiplication by functions of x, translations, differentiations and the operator $[A, x]=Ax-xA$ are studied. An operator A is characterized in terms of the functional equations satisfied by A and also by the properties of the "generatrix function" $g(\lambda, x)=Ae^{\lambda x}/e^{\lambda x}$. The word "metric" should be substituted for the word "norm" on page 5.

F. J. Murray (New York, N. Y.).

Steen, S. W. P. Introduction to the theory of operators. V. Metric rings. Proc. Cambridge Philos. Soc. 36, 139–149 (1940). [MF 1714]

Continuing the development of an abstract theory parallel to that of operator-rings in Hilbert space, the author first completes the discussion of the trace $\tau(A)$ of an abstract operator A, as defined in the fourth paper of the series. In particular the linearity of the trace and its independence of the order of the factors in a product are established. It is then observed that certain rings, termed "doubly bounded," are Hilbert spaces or finite-dimensional complex Euclidean spaces with $\tau(AB^*)$ as the inner product of the elements A and B. Conversely, it is shown that, if a ring with complex multiplication and an adjoint operation, together with a certain algebraic property termed "connectedness," is also a Hilbert space or a finite-dimensional Euclidean space with inner product satisfying the condition $(AC, B)=(A, BC^*)=(C, A^*B)$, then that ring is a doubly-bounded operator-ring as previously defined.

M. H. Stone (Cambridge, Mass.).

Neumark, M. Self-adjoint extensions of the second kind of a symmetric operator. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 53–104 (1940). (Russian. English summary) [MF 1982]

This paper and the one reviewed below present novel investigations and applications of the concept of extension for operators in Hilbert spaces and general complex euclidean spaces. The problem of extension may be formulated thus: to associate with a given operator A_1 in $\mathfrak{H}_1$ a second operator A in $\mathfrak{H}$ so that $\mathfrak{H}_1\subset\mathfrak{H}$ and $A_1\subset A$. This problem is trivial unless the choice of $\mathfrak{H}$ and A is restricted. The case where A_1 is symmetric and the conditions to be observed are that $\mathfrak{H}=\mathfrak{H}_1$ and that A be (maximal) symmetric has been exhaustively studied in the literature. The author is here concerned primarily with the case where A_1 is closed symmetric, A is self-adjoint and $A=A_1$ in $\mathfrak{H}_1$, without the restriction that $\mathfrak{H}=\mathfrak{H}_1$. This case is viewed as a special instance of that where A_1 and A are closed linear operators with domains dense in $\mathfrak{H}_1$ and $\mathfrak{H}$, respectively, and $A=A_1$ in $\mathfrak{H}_1$; under these circumstances, A is said to be an extension of the second kind (of A_1). A full review of methods and results is precluded by the complexity of detail called for in exact statement; it may be said, however, that both methods and results are essentially analogous to those met in other cases. The author gives in theorems 11–15 a complete solution of his primary problem. Since the relation between A and the closed linear manifold $\mathfrak{H}_2=\mathfrak{H}\ominus\mathfrak{H}_1$ may be "pathological" (even the extreme case where A is defined only for the element 0 in $\mathfrak{H}_2$ can be realized in general), the author distinguishes A as a regular extension of the second kind in the favorable case where A is defined on a dense subset of $\mathfrak{H}_2$; and he shows that A_1 always has regular extensions A. As an application, the author shows that every closed symmetric operator H_1 in $\mathfrak{H}_1$ has a spectral resolution $E_1(\lambda)$ and a spectral representation $H_1=\int_{-\infty}^{+\infty}\lambda dE_1(\lambda)$ obtained by putting $E_1(\lambda)=E_1E(\lambda)$, where $E(\lambda)$ is the spectral

resolution for a self-adjoint extension H in $\mathfrak{H}$ (of the second kind) of H_1 and E_1 is the projection of $\mathfrak{H}$ on $\mathfrak{H}_1$; of course, $E_1(\lambda)$ is self-adjoint but need not be a projection. Results like those given in the reviewer's "Linear Transformations in Hilbert Space," Chapter IX, §3, can thus be derived in a simpler and more satisfactory way, as the author shows more fully in a second paper reviewed below. The proofs given in the English version are somewhat less detailed than those presented in the Russian. *M. H. Stone.*

Neumark, M. Spectral functions of a symmetric operator. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 277–318 (1940). (Russian. English summary) [MF 3323]

Here the author continues investigations begun in a previous paper [cf. the preceding review]. As the object of study he takes a family $E_1(\lambda)$, $-\infty<\lambda<+\infty$, of self-adjoint operators (not necessarily projections) in $\mathfrak{H}_1$ which increases monotonely in the wide sense from 0 to 1 in such a way as to be strongly continuous on the left; such a family is a resolution of the identity only when $E_1(\lambda)$ is a projection for each λ. In every sufficiently large space $\mathfrak{H}$ containing $\mathfrak{H}_1$ there exists a resolution of the identity $E(\lambda)$ such that $E_1(\lambda)=E_1E(\lambda)$ in $\mathfrak{H}_1$, where E_1 is the projection of $\mathfrak{H}$ on $\mathfrak{H}_1$; and $E(\lambda)$ is, of course, the spectral resolution of a unique self-adjoint operator H in $\mathfrak{H}$. Moreover each closed symmetric operator H_1 in $\mathfrak{H}$ has associated with it at least one family $E_1(\lambda)$ such that $H_1=\int_{-\infty}^{+\infty}\lambda dE_1(\lambda)$; and, when $E_1(\lambda)$ is represented in the manner indicated above, H is an extension of H_1. That H_1 can be so represented in terms of an arbitrarily chosen self-adjoint extension H of the second kind, the author showed in the previous paper cited; he now obtains the corresponding result for arbitrary self-adjoint extensions of H_1. Further, he shows that every family $E_1(\lambda)$ associated with H_1 can be obtained by this general construction, those which arise from extensions of the second kind being characterized by the property that the domain of H_1 contains those and only those elements f in $\mathfrak{H}_1$ for which $\int_{-\infty}^{+\infty}\lambda^2d(E_1(\lambda)f,f)<+\infty$. By virtue of these facts, the author easily shows that every family $E_1(\lambda)$ associated with H_1 can be approximated "in the sense of Carleman" by a sequence of resolutions of the identity, whose corresponding self-adjoint operators approximate H_1 [see the reviewer's "Linear Transformations in Hilbert Space," Chapter IX, § 3]. There exist families $E_1(\lambda)$ which are associated with no closed symmetric operator H_1. In conclusion, the author defines minimal extensions in a natural way and gives the conditions under which two minimal self-adjoint extensions of a given H_1 determine the same family $E_1(\lambda)$. *M. H. Stone* (Cambridge, Mass.).

Rellich, Franz. Störungstheorie der Spektralzerlegung. IV. Math. Ann. **117**, 356–382 (1940). [MF 3001]

The author continues the investigation of the perturbation of the spectrum of self-adjoint operators $A(\epsilon)$ which depend on a parameter ϵ. In a previous paper [Math. Ann. **116**, 555–570 (1939)] the notion of a "regular" (non-bounded) operator $A(\epsilon)$ was introduced, and it was proved that, for such an operator, power series $\lambda=\lambda_0+\epsilon\lambda_1+\cdots$, $\varphi=\varphi_0+\epsilon\varphi_1+\cdots$ exist which represent eigen-value and eigen-element of $A(\epsilon)$; λ_0 was assumed to be an isolated eigen-value of $A(0)$ of finite multiplicity. In the present paper explicit estimates for the radius of convergence are derived. Further a new criterion for the regularity of $A(\epsilon)$ is formulated; it is based on the assumption that the quadratic form $(u, A(\epsilon)u)$ possesses a power series development while the possibility of developing the operator $A(\epsilon)$ is not required. These results permit a justification of the perturbation procedure and an estimation of the radius of convergence for various differential operators of the Sturm-Liouville and Schrödinger type. *K. Friedrichs.*

Fukamiya, Masanori. On one-parameter groups of operators. Proc. Imp. Acad. Tokyo **16**, 262–265 (1940). [MF 2943]

The author shows that certain of the results of Gelfand [C. R. (Doklady) Akad. Sci. URSS (N.S.) **25**, 713–718 (1939); these Rev. **1**, 338] can be obtained by the simpler methods of Stone if the space is separable. Results concerning the resolvent are carried farther than in Gelfand's paper, but the real and imaginary axes have been interchanged through the omission of a factor i. *R. H. Cameron.*

Kakutani, Shizuo. On the uniform ergodic theorem concerning real linear operations. Jap. J. Math. **17**, 5–12 (1940). [MF 3110]

The author wishes to withdraw this paper, due to an error in the principal theorem as stated.
S. Kakutani (Princeton, N. J.).

Yosida, Kôsaku. Ergodic theorems of Birkhoff-Khintchine's type. Jap. J. Math. **17**, 31–36 (1940). [MF 3155]

Let T be a linear operator in the space L^p and define $f^{(n)}=T^nf$. Then if $\|T^n\|$ is bounded and

$$\overline{\lim_{n\to\infty}}\left|\frac{1}{n}\sum_{m=1}^{n}f^{(m)}(t)\right|<\infty$$

almost everywhere for every $f\epsilon L^p$ we have, in case $p>1$, for every $f\epsilon L^p$ an $f^*\epsilon L^p$ with $Tf^*=f^*$ and

$$\frac{1}{n}\sum_{m=1}^{n}f^{(m)}(t)\to f^*(t)$$

almost everywhere as well as in the mean. In case $p=1$ such an f^* exists for those $f\epsilon L$ for which $\lim_n f^{(n)}(t)/n=0$ almost everywhere and for which also the sequence $(1/n)\sum_{m=1}^{n}f^{(m)}$ contains a subsequence weakly convergent. This represents an advance in the strong ergodic theory of G. D. Birkhoff in that T is not assumed to have an inverse nor to be generated by a measure preserving automorphism. The essential part of the proof is the following lemma. The operation $\bar{T}f=\bar{f}$,

$$\bar{f}(t)=\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{m=1}^{n}f^{(m)}(t)-\underline{\lim_{n\to\infty}}\sum_{m=1}^{n}f^{(m)}(t),$$

is continuous when considered as a function on L^p to the Fréchet space of measurable functions. *N. Dunford.*

Yosida, Kôsaku. An abstract treatment of the individual ergodic theorem. Proc. Imp. Acad. Tokyo **16**, 280–284 (1940). [MF 2708]

The results of the preceding review are given in a more abstract form. The method of proof remains the same. The space of measurable functions used in the lemma is replaced by a linear metric lattice of type F in which (i) sup (x, y), inf (x, y) exist as continuous functions, (ii) sequences which are bounded (with respect to order) have least upper bounds, (iii) $x_n\to x$ if

$$\inf_n \sup_{m\geq n} x_m=\sup_n \inf_{m\geq n} x_m=x,$$

and (iv) $x\geqq y\geqq -x$ implies $\|x\|\geqq\|y\|$. *N. Dunford.*

Hilmy, Heinrich. Sur le théorème ergodique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 213–216 (1939). [MF 2904]

The author proves a known generalization of the ergodic theorem of G. D. Birkhoff concerning one parameter groups of measure preserving transformations of a space M, from the case meas $(M)<\infty$ to the case meas $(M)=\infty$. More general results were given by N. Wiener [Duke Math. J. **5**, 1–18 (1939)] and K. Yosida-S. Kakutani [Proc. Imp. Acad. Tokyo **15**, 165–168 (1939); these Rev. **1**, 59].

S. Kakutani (Princeton, N. J.).

Bebutoff, M. et Stepanoff, W. Sur le changement du temps dans les systèmes dynamiques possédant une mesure invariante. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 217–219 (1939). [MF 2905]

Let $f_0(p, t)$, $p\epsilon R$, $-\infty<t<\infty$, be a one-parameter group of automorphisms of a separable metric space R, the function $f_0(p, t)$ being continuous in p and t. Let $f_1(p, t')$ be another such group having the same streamlines, that is, such that for each p there is a $1:1$ correspondence of values t and t' such that $f_0(p, t)=f_1(p, t')$. It is shown that if the group f_0 admits a locally finite invariant measure μ then f_1 also admits such a measure μ^*, and that μ^* is positive for any open set for which μ is positive. The measure μ^* is defined from μ by means of local sets of section in the flow. The result permits the generalization to separable metric spaces of the reasoning by which the second author [Compositio Math. **3**, 239–253 (1936)] obtained an extension of the ergodic theorem to infinite measures in the metrically transitive case, a result since superseded by a more general formulation due to E. Hopf [Ergodentheorie, Ergebnisse der Mathematik, vol. 3, Berlin, 1937, § 14].

J. C. Oxtoby (Bryn Mawr, Pa.).

Hopf, Eberhard. Statistik der Lösungen geodätischer Probleme vom unstabilen Typus. II. Math. Ann. **117**, 590–608 (1940). [MF 3099]

In a previous paper (I) with the same title [Ber. Verh. Sächs. Akad. Wiss. Leipzig **91**, 261–304 (1939); these Rev. **1**, 243] the author showed that the geodesic flows on certain classes of surfaces are either ergodic or dissipative. Among other conditions imposed it was assumed that the Gaussian curvature of the surface lay between two negative constants. In the present paper it is shown that this condition may be replaced by a uniform instability condition on the equations of variation of the geodesics. Specifically, there exist positive constants a and b such that, if $y(s)$ is the solution of the Jacobi equation of any geodesic, s the arc-length along the geodesic, such that $y(0)=0$, $y'(0)=1$, then

$$y(s_2)>ae^{b(s_2-s_1)}y(s_1), \qquad s_2>s_1>0.$$

The proofs follow the pattern of (I). In addition, a criterion for the fulfillment of the desired conditions is derived in terms of conditions on the regions on the surface in which the Gaussian curvature is positive or negative. As the author states, it would be desirable to obtain more definitive conditions of this kind.

G. A. Hedlund.

Theory of Probability

Koopman, B. O. The bases of probability. Bull. Amer. Math. Soc. **46**, 763–774 (1940). [MF 2921]

The concepts of a previous paper on the axioms and algebra of intuitive probability [Ann. of Math. (2) **41**, 269–292 (1940); these Rev. **1**, 245] are discussed. Further details will appear in a forthcoming paper in the Annals of Mathematics. An interesting development beyond the earlier paper is given, by which, assuming various hypotheses on a set of experiments, the intuitive probability becomes equal to the limit of success ratios.

J. L. Doob (Urbana, Ill.).

Birnbaum, Z. W. and Zuckerman, Herbert S. On the properties of a collective. Amer. J. Math. **62**, 787–791 (1940). [MF 2879]

Let $a_1, a_2, \cdots, x_1, x_2, \cdots$ be sequences of numbers with $a_j, x_j=1$ or 0. Let $t=\sum_1^\infty x_j2^{-j}$. Then Lebesgue t-measure defines a measure on the space of sequences $\{x_j\}$. The authors prove that $(\sum_1^n a_jx_j)/(\sum_1^n x_j)$ has the same set of limiting values as $(\sum_1^n a_j)/n$ when $n\rightarrow\infty$, for almost all sequences $\{x_j\}$. [A special case is due to Steinhaus, Fund. Math. **4**, 305 (1923).] [Note by reviewer: the result follows directly from the strong law of large numbers, in accordance with which, if $\sum_1^\infty a_j=\infty$, it follows that $(\sum_1^n a_jx_j)/(\sum_1^n a_j)\rightarrow\frac{1}{2}$ and $(\sum_1^n x_j)/n\rightarrow\frac{1}{2}$ for almost all $\{x_j\}$.]

J. L. Doob (Urbana, Ill.).

Garti, Y. Les lois de probabilité pour les fonctions statistiques (cas de collectifs à plusieurs dimensions). Rev. Math. Union Interbalkan. **3**, 21–39 (1940). [MF 2674]

In this paper the central limit theorem is generalized in the following manner. Let $S_n(X)$ be the function of the points $X_1, X_2, \cdots, X_n$, and X of k-space such that $n\cdot S_n(X)$ is the number of values of μ for which $X_\mu\leqq X$, where $X_\mu\leqq X$ indicates that each coordinate $x_i^{(\mu)}$ of X_μ is less than or equal to the corresponding coordinate x_i of X. Then any functional $f\{S_n(X)\}$ of $S_n(X)$ is called a statistical function of the points $X_1, \cdots, X_n$. Thus the k-dimensional moments, the correlation coefficient, etc., are statistical functions. Next let these points be interpreted as fortuitous variables, let $V_\mu'(X)$ be the probability that the inequality $X_\mu\leqq X$ will hold, let $V_n(X)=1/n[V_1'(X)+V_2'(X)+\cdots+V_n'(X)]$ and let H_n be a number depending on n but not on X. Then $H_n[f\{S_n(X)\}-f\{V_n(X)\}]$ is a one dimensional fortuitous variable. The author gives sufficient conditions that the distribution of this variable approaches the Gaussian as n becomes infinite. The methods used in this paper were developed by von Mises for the one-dimensional case. The author has generalized these methods to the case of k dimensions.

A. H. Copeland (Ann Arbor, Mich.).

Seitz, Boris. Sur une équation diophantienne en rapport avec le calcul des probabilités. Comment. Math. Helv. **12**, 323–325 (1940). [MF 3061]

There are n urns containing white balls and black balls in various compositions. An urn is selected at random and a ball is drawn from it. The ball is then returned and all of the balls are emptied into a single urn. A ball is drawn at random from this urn. It is desired to find the possible compositions of the urns so that the above two modes of drawing will yield the same probability of obtaining a white ball. The author obtains an infinite system of solutions of the resulting diophantine equation.

A. H. Copeland.

Chung, Kai-Lai. Sur un théorème de M. Gumbel. C. R. Acad. Sci. Paris **210**, 620–621 (1940). [MF 3035]

In this paper it is shown that

$$p_m(1, 2, \cdots, n)\leqq\frac{1}{C_{n-m}^{k-m}}\sum p_m(\nu_1, \nu_2, \cdots, \nu_k),$$

where $p_m(\nu_1, \nu_2, \cdots, \nu_k)$ is the probability that at least m of the events $E_{\nu_1}, E_{\nu_2}, \cdots, E_{\nu_k}$ will occur, and where the summation is extended over all possible groups of k events selected from the events $E_1, E_2, \cdots, E_n$. The case $m=1$ gives a theorem due to Gumbel. The author obtains a simplification of Gumbel's proof and then demonstrates the above extension of Gumbel's theorem.

A. H. Copeland (Ann Arbor, Mich.).

Riebesell, Paul. Neue deutsche Forschungen über das Gesetz der grossen Zahl. Bl. Versich.-Math. **5**, 68–75 (1940). [MF 2759]

Hartman, Philip and Wintner, Aurel. On the spherical approach to the normal distribution law. Amer. J. Math. **62**, 759–779 (1940). [MF 2877]

The authors consider probability distributions in the n-dimensional space of points $(x_1, \cdots, x_n)$ such that the measure assigned to a set is invariant under rotations about the origin. Such a distribution determines a distribution function (d.f.) $\sigma(x) = \text{prob.}\ \{x_1 < x\}$, and σ is then said to be of class Ω if $\sigma(x)$ can be obtained in this way for all n. (I) σ is of class Ω if and only if there is a d.f. $\tau(t)$, with $\tau(0)=0$, such that (1) $\sigma(x)=\int_0^\infty \sigma^*(x/t)d\tau(t)$, where $\sigma^*(x)=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^x e^{-y^2/2}dy$. (II) A general theorem is proved which implies that if $\omega(x)$ is of class Ω: $\omega(x)=\int_0^\infty \sigma^*(x/t)d\tau_\omega(t)$ then a d.f. $\sigma(x)$ of class Ω can be represented by (1) with ω substituted for σ^* if and only if there is a d.f. $\xi(t)$, with $\xi(0)=0$, such that $\tau(t)=\int_0^\infty \tau_\omega(t/y)d\xi(y)$.. Moreover every d.f. of class Ω can be represented by (1) with ω substituted for σ^* if and only if there is a positive T such that $\omega(x/T) = \sigma^*(x)$ for all x. (III) The results are applied to stable d.f., known to be of class Ω. (IV) If an n-dimensional probability distribution, $n \geqq 2$, not concentrated at a single point, has the property that there is a k, $0<k<n$, such that the probability distributions projected on any orthogonal $n-k$, k dimensional pair of hyperplanes are independent, then the original probability distribution is itself an n-dimensional Gaussian distribution, and $x_1, \cdots, x_n$ are independent with the same variance. [Cf. also Geppert, Giorn. Ist. Ital. Attuari **7**, 378–391 (1936), and Kac, Amer. J. Math. **61**, 726–728 (1939); these Rev. **1**, 62, who obtain somewhat more special results.] (V) The projections on a coordinate axis of the equidistribution over the $(x_1, \cdots, x_n)$-set $\{\sum_1^n |x_j|^p \leqq r^p\}$ is determined and it is shown that, if $r=n^{1/p}$, there is a limiting distribution (as $n\to\infty$) on the axis, with density proportional to $\exp(-|x|^p/p)$. *J. L. Doob.*

Bernstein, S. Nouvelles applications des grandeurs aléatoires presqu'indépendantes. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 137–150 (1940). (Russian. French summary) [MF 2733]

Let $u_1, u_2, \cdots$ be a sequence of mutually dependent chance variables. The author generalizes and puts into more usable form his well-known conditions [Math. Ann. **97**, 1-59 (1927); cf. also P. Lévy, Bull. Sci. Math. (2) **59**, 109-128 (1935)] that $u_1+\cdots+u_n$ have a nearly normal distribution for large n. Let a given urn contain a white and b black balls. Drawings are made: if a white ball is drawn, it is replaced and R balls are added, ν white and $R-\nu$ black; if a black ball is drawn, it is replaced and R balls are added, ν_1 white and $R-\nu_1$ black. Suppose $(R-\nu)\nu_1(\nu-\nu_1)\neq 0$ and $2(\nu-\nu_1)\leqq R$. Then (applying the above results) if x_n is the number of white balls after the nth drawing, x_n is nearly normally distributed for large n. Suppose $E(u_n)$ (expectation of u_n) is 0, and let $B_n=E(u_1+\cdots+u_n)^2$. Suppose $u_1, \cdots, u_{n-1}$ are given values. Let the conditional expectation of u_n be a_n; let the conditional dispersion of u_n be b_n; let the l.u.b. of the conditional expectation of $|u_n|^{2+\delta}$ be $c_n (\delta>0)$. Then if $\sum_{i=1}^n c_i B_n^{-1-\delta/2}\to 0$, and if $E(\sum_1^n a_i)^2/B_n\to 0$, it is proved that $u_1+\cdots+u_n$ is nearly normally distributed for large n (with dispersion B_n) if and only if $\sum_1^n b_i/B_n\to 1$ in probability. The general form of the limiting law if the latter condition is not satisfied is also found. The results are all obtained using characteristic functions.

J. L. Doob (Urbana, Ill.).

Kawata, Tatsuo. On the division of a probability law. Proc. Imp. Acad. Tokyo **16**, 249–254 (1940). [MF 2941]

It is well known [cf. e.g. P. Lévy, Addition des variables aléatoires, Paris, 1937, p. 189] that the division of distribution functions is not uniquely determined, that is to say, that there exist distribution functions $F(x), \cdots, F_3(x)$ such that (*) $F(x)=F_1(x)*F_2(x)=F_1(x)*F_3(x)$, where $F_2(x)\not\equiv F_3(x)$. It is easily seen that this implies the vanishing in some interval of the characteristic functions of $F_1(x)$, and of $F(x)$. Hence the author's previous results [the same Proc. **16**, 157–160 (1940); these Rev. **1**, 330] on the non-vanishing of characteristic functions contain sufficient conditions for the uniqueness of the factorization of $F(x)$. In particular, if for some constant $a>0$ we have (⁑) $F(-x+a)-F(-x-a) = O(\exp(-\theta(x)))$ as $x\to\infty$, where $\theta(x)$ is positive, non-decreasing and such that $\int_1^\infty (\theta(x)/x^2)dx$ diverges, the (*) cannot hold. It is now shown that if the last integral is convergent there always exists a distribution function $F(x)$ satisfying (⁑) and such that (*) holds. *W. Feller.*

Doeblin, W. Remarques sur la théorie métrique des fractions continues. Compositio Math. **7**, 353–371 (1940). [MF 1869]

The author proves by using the methods of the theory of probability some of the more important theorems on the metric properties of continued fractions of Borel, Kuzmin, P. Levy, Khintchine and Denjoy. *P. Erdös.*

McCrea, W. H. and Whipple, F. J. W. Random paths in two and three dimensions. Proc. Roy. Soc. Edinburgh **60**, 281–298 (1940). [MF 3008]

Consider a rectangular lattice in 2 or 3 dimensions and a particle P moving in the lattice-domain Γ in such a way that when P is at any particular point M of Γ it is equally likely to move to any of the four neighboring points of M. On arrival of P at a boundary point the movement ceases. P is liberated at the interior point N of D. The authors are concerned with the expectation $F(M;N)$ that P will visit the interior point M of D. It is known that $F(M;N)$ can be interpreted as Green's function for a boundary-value problem for a system of linear equations corresponding to the Dirichlet problem for harmonic functions. The present paper gives explicit solutions in terms of trigonometric solutions; finite rectangles in the plane and their generalizations (such as strips) in two and three dimensions are considered. Finally the authors carefully consider passages to the limit for the whole plane and space; the asymptotic behavior of $F(M;N)$ is different in the case of 2 and 3 dimensions.

W. Feller (Providence, R. I.).

Lévy, Paul. Le mouvement brownien plan. Amer. J. Math. **62**, 487–550 (1940). [MF 2454]

The author continues the researches of previous papers

[Compositio Math. **7**, 283–339 (1939), these Rev. **1**, 150; C. R. Acad. Sci. Paris **209**, 140–142, 387 (1939), these Rev. **1**, 22] on a random function $X(t)$ whose increments on non-overlapping t-intervals are independent chance variables, $X(t+h)-X(t)$ having a Gaussian distribution with mean 0 and mean square deviation h. It is supposed that $X(0)=0$. The following results are characteristic of those obtained. (1) If $\xi(t)=X(t)/\sqrt{t}$, the one-parameter family of chance variables $\{\xi(t)\}$ is stochastically the same as the family $\{\xi(1/t)\}$, so that a correspondence is set up between theorems for t near 0 and theorems for t large. (2) $X(t)$ vanishes on a t-set having 0 as a limit point (with probability 1) and (with probability 1) the set of zeros contains no point isolated on both right and left. (3) The limiting frequency of the values of n for which $X(t)$ has a root in the interval (q^{n+1}, q^n) (where $0<q<1$) is $2/\pi \arctan ((1-q)/q)^{\frac{1}{2}}$. (4) If $t_1, t_2, \cdots$ is a sequence of numbers in $0<t<1$, dense in that interval, then $B_n=\sum_1^{n-1} X(t_{j+1})-X(t_j)\to 1$, with probability 1. If the t_j are chosen at random, with $t_1, t_2, \cdots$ independent chance variables, and Prob. $(a<t_j<b)=b-a$, then $B_n\to 1$, with probability 1. Almost every $X(t)$ has the property that if $t_1, t_2, \cdots$ are chosen as just explained, $B_n\to 1$, with probability 1. Explicit examples of functions with this property are given.

If $X(t)$, $Y(t)$ are independent random functions, each with the distribution of $X(t)$, they determine parametrically a continuous random curve C. (5) If $\theta(s)$ is the angular increase on C (as seen from the origin) from $t=1$ to $t=s$, the second moment of $\theta(s)$ is infinite. (6) If C' is the arc of C for which $t'\leqq t\leqq t''$, the bounding curve Γ of the smallest convex body including C' is determined (with probability 1) by a denumerable number of lines, each tangent to C in two points; Γ has a tangent at every point, and the angle this tangent makes with a given direction is a continuous function of the arc length on Γ. (7) Let $S(t')$ be the area between the curve C for $0\leqq t\leqq t'$ and the chord joining the endpoints, and let $L(t)$ be the length of this chord. Then the combined distribution function of $S(t)$, $L(t)$ is shown to satisfy a certain partial differential equation. (8) The curve C has planar measure 0. (9) Some of the above results are compared with corresponding results, true for curves having arcs similar to larger arcs, or true for random curves, not necessarily having the distributions of those above, but having certain subarcs stochastically similar to larger arcs. [Cf. earlier papers by the author, J. École Polytech. (3) **144**, 227–247, 249–291 (1938); C. R. Acad. Sci. Paris **207**, 1152–1154 (1938).] *J. L. Doob* (Urbana, Ill.).

Doob, J. L. and Ambrose, W. On two formulations of the theory of stochastic processes depending upon a continuous parameter. Ann. of Math. (2) **41**, 737–745 (1940). [MF 3015]

The formulations referred to in the title are those of Doob and of Wiener. The first one was described in these Rev. **1**, 343. Wiener's approach is described by the authors as follows: he takes a function $f(t, x)$, subject to certain conditions, and then considers the collection of t-functions obtained by fixing x and allowing t to vary; the measure on this space of t-functions is then defined in terms of a measure on x-space. In the present paper it is shown that both approaches are equivalent. Furthermore conditions are given that $f(t, x)$ defines a measurable stochastic process in Doob's sense. *W. Feller* (Providence, R. I.).

Theoretical Statistics

***Dahlberg, Gunnar. Statistical Methods for Medical and Biological Students.** George Allen & Unwin, Ltd., London; Interscience Publishers, Inc., New York, 1940. 232 pp. $2.75.

Contents: I. Probability. II. Combinations. III. Compound probability and the binomial theorem. IV. Statistical characterization of material. V. Quantitative statistics (characteristics of orientation, measures of dispersion). VI. Smoothing distributions. VII. Standard error of statistical characteristics. VIII. Standard error of products and quotients. IX. Comparisons between materials. X. Addition of materials and groups of factors. XI. Variability. XII. Errors of estimation. XIII. The personal variability. XIV. Qualitative statistics. XV. Significant differences; influence of sex and age. XVI. Goodness of fit. XVII. Defects of materials; complication statistics. XVIII. Measures of asymmetry and dissymmetry. XIX. Correlation. XX. On the conception of normalcy. XXI. Medicine as a science and an art. Appendix I. $(n-1)$-correction; derivation of the formula of standard error. Appendix II. Table of the probability integral; values of χ^2 at different numbers of degrees of freedom (n); table of trigonometric functions to be used in calculating tetrachoric r; values of χ_r to be used in calculating the standard error of tetrachoric r.

***Two Papers by Bayes.** Prepared under the direction of W. Edwards Deming. Department of Agriculture, Washington, D. C., 1940. xvi+52 pp.

This book contains the following two papers by Thomas Bayes: (1) "An essay toward solving a problem in the doctrine of chances, with Richard Price's foreword and discussion; Phil. Trans. Royal Soc., pp. 370–418, 1763," with a commentary by Edward C. Molina; (2) "A letter on asymptotic series from Bayes to John Canton; pp. 269–271 of the same volume," with a commentary by W. Edwards Deming.

***Sheppard, W. F. The Probability Integral.** British Association for the Advancement of Science, Mathematical Tables, vol. 7. Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1939. xi+34 pp. $2.50.

Put

$$\varphi(x)=\frac{1}{(2\pi)^{\frac{1}{2}}}e^{-x^2/2}, \quad \Phi(x)=\int_x^\infty \varphi(t)dt, \quad F(x)=\frac{\Phi(x)}{\varphi(x)}.$$

The present long-awaited book contains the following tables, all for the interval $0\leqq x\leqq 10$. (i) Values $F(x)$ and its reduced derivatives (that is, $(h^n/n!)F^{(n)}(x)$) at intervals of .01 to 12 decimal places. (ii) The same functions at intervals of .1 to 24 decimal places. (iii) $-\log_e \Phi(x)$ and its reduced derivatives at intervals of .1 to 16 decimal places and for integral values of x to 24 decimal places. (iv) $\log_{10} \Phi(x)$ and its reduced derivatives at intervals of .1 to 12 decimal places. (v) $\log_{10} \Phi(x)$, with second central differences, at intervals of .01 to 8 decimal places. *W. Feller.*

Wald, Abraham. The fitting of straight lines if both variables are subject to error. Ann. Math. Statistics **11**, 285–300 (1940). [MF 2833]

The author summarizes effectively the results to date on the problem stated in the title, and offers as solution a method that may be applied without a priori assumptions

regarding the standard deviations of the errors. The proposed procedure in brief is to separate the data into two groups. The suggested method of separation is unsymmetric as between the two variables and might raise further questions in the mind of the reader. *A. A. Bennett.*

Singleton, Robert R. A method for minimizing the sum of absolute values of deviations. Ann. Math. Statistics **11**, 301–310 (1940). [MF 2834]

Rhodes [Philos. Mag. (7) **9**, 974–992 (1930)] described a recursive and iterative method of fitting linear regression equations by minimizing the sum of the absolute values of the residuals. In the present paper the author derives a method which is free of the recursive feature and discusses its theory by means of a geometric analogue. The method proceeds, like Rhodes, by finding successive restricted minima but utilizes steepest descents. It is shown that a unique minimum exists and a criterion is given for recognizing it when it is reached. An illustrative numerical example of Rhodes is reworked by the new method for comparison. *C. C. Craig* (Ann Arbor, Mich.).

Andersson, Walter. A general formula for the normal mean errors of the coefficients in parabolic least squares graduation. Skand. Aktuarietidskr. **1940**, 44–53 (1940). [MF 2963]

Dwyer, P. S. Combinatorial formulas for the rth standard moment of the sample sum, of the sample mean, and of the normal curve. Ann. Math. Statistics **11**, 353–355 (1940). [MF 2838]

"The rth standard moment of the normal curve is equal to the number of ways in which r things can be grouped in pairs." That is: $\alpha_{2s}=(2s!)/2^s s!$; $\alpha_{2s+1}=0$ with $r=2s$ or $2s+1$. The author makes use of combinatorial analysis as developed in his papers [Ann. Math. Statistics **8**, 21–65 (1937); **9**, 1–47, 97–132 (1938)] to obtain the above values of α_{2s} and α_{2s+1} as limiting values for $n\rightarrow\infty$, where n is the number of observations in a sample from an infinite population. He starts with moments of the sample sum. These involve coefficients which are modified forms of the multinomial coefficients. *E. L. Dodd* (Austin, Tex.).

Pierce, Joseph A. A study of a universe of n finite populations with application to moment-function adjustments for grouped data. Ann. Math. Statistics **11**, 311–334 (1940). [MF 2835]

In this paper the author derives general formulas for mean values of sample moments in samples of size n formed by drawing each element from a different finite population. He verifies that under the appropriate conditions these general formulas reduce to various known formulas such as those for repeated sampling from a single finite population, those for sampling from a single infinite population, etc. He then applies his results to the problem of determining the mean values of moments from samples of grouped data, deriving formulas for correcting bias (due to grouping) of central moments and Thiele semi-invariants.
S. S. Wilks (Princeton, N. J.).

Truksa, L. The simultaneous distribution in samples of mean and standard deviation, and of mean and variance. Biometrika **31**, 256–271 (1940). [MF 2266]

Let $f(x)$ be the probability density function of a random variable x. Denote by $\bar{x}$ the arithmetic mean and by s the standard deviation of a sample of t independent observations on x, that is,

$$\bar{x}=\frac{1}{t}\sum_{\alpha=1}^{t} x_\alpha,\quad s^2=\frac{1}{t}\sum_{\alpha=1}^{t}(x_\alpha-\bar{x})^2.$$

By an extension of this sample of size t to one of size $t+2$, we get a sample, the mean of which is

$$\bar{X}=\frac{1}{t+2}\sum_{1}^{t+2} x_\alpha,$$

and the standard deviation S is given by

$$S^2=\frac{1}{t+2}\sum_{1}^{t+2}(x_\alpha-\bar{X})^2.$$

Denote by $F_t(\bar{x}, s)$ the joint probability density function of $\bar{x}$ and s, and let $F_{t+2}(\bar{X}, S)$ be the joint probability density function of $\bar{X}$ and S. Calculating the conditional probability distribution of $\bar{X}$ and S when $\bar{x}$ and s are fixed, multiplying it by $F_t(\bar{x}, s)$ and integrating it with respect to $\bar{x}$ and s, the author obtains the following recurrence relationship:

$$F_{t+2}(\bar{X}, S)=2S(t+2)^2\int_\Omega\int F_t(\bar{x}, s)f\left(\frac{t+2}{2}\bar{X}-\frac{t}{2}\bar{x}+\frac{1}{2}\alpha\right)\times f\left(\frac{t+2}{2}\bar{X}-\frac{t}{2}\bar{x}-\frac{1}{2}\alpha\right)\frac{d\bar{x}ds}{\alpha},$$

where

$$\alpha=(\{2[(t+2)S^2-ts^2]-t(t+2)(\bar{X}-\bar{x})^2\})^{\frac{1}{2}}.$$

The limits of integration in the above recurrence formula are discussed and the functions $F_1(\bar{X}, S)$ and $F_2(\bar{X}, S)$ are explicitly calculated. The application of the recurrence formula to the calculation of $F_t(\bar{x}, s)$ is illustrated by some examples. *A. Wald* (New York, N. Y.).

Olmstead, P. S. Note on theoretical and observed distributions of repetitive occurrences. Ann. Math. Statistics **11**, 363–366 (1940). [MF 2841]

Given a repetitive operation with constant probability p of producing an event E, a method is provided for identifying significantly long or short lengths for individual runs of the event in a sequence of repetitions and significantly high or low average lengths for groups of several runs. Applications are cited. *W. A. Shewhart.*

Hsu, P. L. An algebraic derivation of the distribution of rectangular coordinates. Proc. Edinburgh Math. Soc. (2) **6**, 185–189 (1940). [MF 2854]

Given the random variables x_{ir} $(i=1, \cdots, q; r=1, \cdots, m; q\leqq m)$, with $s_{ij}=\sum_{r=1}^{m}x_{ir}x_{jr}$, it is assumed that the elementary frequency law $p(x_{11}, \cdots, x_{qm})$ is explicitly a function of the s_{ij}'s above. Generalized rectangular coordinates t_{ij} are defined by the equation: $S=TT'$ in which S is the matrix of s_{ij}'s and T that of the t_{ij}'s with $t_{ij}=0$ for $i<j$ and $t_{ii}\geqq 0$. The elementary probability law of the t_{ij}'s is derived in a purely algebraic manner. [A geometrical derivation in the case that $p(x_{11}, \cdots, x_{qm})$ is normal was given by Mahalanobis, Bose and Roy, Sankhya **3**, 1–40 (1937).] From this another step, using transformations given in the paper, gives the distribution of the product moments s_{ij}, so that the well-known result in the case of samples from normal is a special case of this one. *C. C. Craig* (Ann Arbor, Mich.).

Kendall, M. G. Proof of Fisher's rules for ascertaining the sampling semi-invariants of k-statistics. Ann. Eugenics 10, 215–222 (1940). [MF 2575]

R. A. Fisher's k-statistic k_p of order p is defined as the symmetric and homogeneous polynomial of degree p in the sample values $x_1, x_2, x_3, \cdots, x_n$, whose mean value is κ_p, the pth semi-invariant of the parent distribution. Fisher's rules for obtaining the sampling semi-invariants of the k-statistics were somewhat heuristically derived originally. The author gives a systematic proof of the validity of Fisher's rules, making use of some differential operators and simple extensions which he (Kendall) used in a previous paper for investigating properties of k-statistics [Ann. Eugenics 10, 106–111 (1940); these Rev. 1, 347].

S. S. Wilks (Princeton, N. J.).

Brown, George W. Reduction of a certain class of composite statistical hypotheses. Ann. Math. Statistics 11, 254–270 (1940). [MF 2831]

The purpose of this paper is to show that, under appropriate conditions, the knowledge of the sample distribution on some manifolds determines the original distribution except for location or scale parameters. Let $x_1, \cdots, x_n$ be mutually independent random variables with the same distribution function $F(x)$. Let it be known that $y_1, \cdots, y_n$ are independent random variables having the same distribution function $G(x)$, and that the joint $(n-1)$-dimensional distribution of $(y_1-y_n, \cdots, y_{n-1}-y_n)$ is the same as that of $(x_1-x_n, \cdots, x_{n-1}-x_n)$. It is then shown that, under a mild restriction as to the zeros of the characteristic function of $F(x)$, there is a constant θ such that $F(x)=G(x-\theta)$. Similarly, if $(y_1/y_n, \cdots, y_{n-1}/y_n)$ has the same joint distribution as $(x_1/x_n, \cdots, x_{n-1}/x_n)$, then $F(x)=G(x\theta)$. Various generalizations are proved: in particular x_a-x_n and x_a/x_n are replaced by more general linear functions, and the case of more-dimensional random variables is also treated. Applications to the problem of testing statistical hypothesis are indicated.

W. Feller (Providence, R. I.).

Dunlap, Jack W. Note on the computation of tetrachoric correlation. Psychometrika 5, 137–140 (1940). [MF 3159]

Babitz, Milton and Keys, Noel. A method for approximating the average intercorrelation coefficient by correlating the parts with the sum of the parts. Psychometrika 5, 283–288 (1940). [MF 3317]

Woodbury, Max A. Rank correlation when there are equal variates. Ann. Math. Statistics 11, 358–362 (1940). [MF 2840]

Provides a method with tables for computing the average of the values of the rank correlation coefficient corresponding to all possible ways of assigning ranks within groups of equal values of a variate.

W. A. Shewhart.

Masuyama, Motosaburo. On the meaning of the symmetric correlation coefficient between vector sets. Proc. Phys.-Math. Soc. Japan (3) 22, 579–585 (1940). [MF 2697]

[Continuation of a paper in the same Proc. 21, 638–647 (1939). Cf. these Rev. 1, 151; the name of the author of this paper was misprinted as Masuamya in the journal, and the misprint also appears in the Reviews.]

In the first paper correlation coefficients between two sets of 3-vectors were discussed. In the present paper it is a question of m sets of n-dimensional vectors. First a correlation tensor between two sets of vectors is defined, and from this the author is led to the definition of a multiple correlation tensor between a single set of vectors and m sets of vectors. Correlation coefficients are defined as the square roots of the determinants of the corresponding correlation tensors.

J. L. Synge (Toronto, Ont.).

Dressel, Paul L. Some remarks on the Kuder-Richardson reliability coefficient. Psychometrika 5, 305–310 (1940). [MF 3319]

This gives a derivation under somewhat less restrictive conditions of a reliability coefficient given by G. F. Kuder and M. W. Richardson [Psychometrika 2, 151–160 (1937)]. It is shown that the coefficient may be used in connection with the Spearman-Brown prediction formula and various special forms of the coefficient are derived for different ways of scoring test results.

C. C. Craig (Ann Arbor, Mich.).

Lawley, D. N. The estimation of factor loadings by the method of maximum likelihood. Proc. Roy. Soc. Edinburgh 60, 64–82 (1940). [MF 2179]

With the usual set-up of linear regression equations used in factor analysis, the matrix equations are obtained in a quite simple form for the estimation of the factor loadings from observed test scores by the method of maximum likelihood. The work is carried out in the case that there are two general factors present but the results are general in form. The matrix equations naturally do not have a unique set of solutions due to the freedom in the space of general factors, and a unique set is picked out by the use of the largest latent roots of a pair of matrices associated with the equations of condition, which gives the method a certain resemblance to the method of principal components due to Hotelling. It is noted that the results are independent of the scale of measurement of the test scores. To obtain numerical solutions an iterative method is proposed and illustrated (in a case where only a single factor is assumed) though no attempt is made to show that the process is convergent. Finally a test of significance for factor loadings in large samples obtained in this way is developed and illustrated.

C. C. Craig (Ann Arbor, Mich.).

Young, Gale and Householder, A. S. Factorial invariance and significance. Psychometrika 5, 47–56 (1940). [MF 3160]

This brief paper presents in clear style a critical summary of various attempts at factor determination and offers the thesis that the indeterminacy in the matrix of factor loadings is fully taken care of by invariance requirements, the invariance principle being "that any consistent system of factor analysis must yield the same values (apart from errors) for the factor loadings of individuals or tests, regardless of the combinations in which they are presented." These considerations argue against the use of specific factors, normal scores and such asymmetric procedures as give rise to a distinction between direct and inverse factor problems. Thurstone's suggestion of looking for "simple structures" is about the only answer hitherto made to the problem of finding "meaningful" directions for the primary factor axes. Eventually the experimenter must be interested in factors whose variation from one experiment to another can be directly studied and perhaps controlled, such as growth laws in the population matrix.

A. A. Bennett.

Hotelling, Harold. The selection of variates for use in prediction with some comments on the general problem of nuisance parameters. Ann. Math. Statistics 11, 271–283 (1940). [MF 2832]

A problem in mathematical statistics not yet satisfactorily solved is that of measuring the effect of the selection which is involved in choosing from among a set of r independent variables the subset of k variables which when used in a linear regression function yield the highest multiple correlation with a given dependent variable. The problem of measuring the effect of such selection reduces to a problem of developing appropriate significance tests. The author considers several special cases of this problem. In particular, he obtains the solution for the case in which it is desired to select one of two independent variables, assuming the independent variables to be "fixed" and assuming the dependent variable to be normally distributed about a linear regression function of the two variables. He extends the solution of this problem to the case in which it is desired to test the hypothesis that all of the independent variables in a set have equal predictive power for a dependent variable y, and also to the case in which a set of independent variables have already been selected and in which it is desired to test the hypothesis that each in an additional set of variables has equal additional predictive power with the dependent variable y. Criteria for testing these two hypotheses are derived. Some remarks are made on the problem of isolating parameters in the problem of estimation, that is, of devising statistical functions to estimate a given population parameter without involving irrelevant population parameters.

S. S. Wilks (Princeton, N. J.).

Cochran, W. G. The analysis of variance when experimental errors follow the Poisson or binomial laws. Ann. Math. Statistics 11, 335–347 (1940). [MF 2836]

It is fairly well known that, if x is a random variable with a Poisson distribution law, then in large samples of size n the distribution of $2\sqrt{\bar{x}}$, where $\bar{x}$ is the sample mean, is approximately normal with zero mean and variance $1/n$. Similarly, in the case of samples from a binomial population, the distribution of $2\sin^{-1}\sqrt{\bar{p}}$, where $\bar{p}$ is the relative frequency of "successes" in samples of size n, is approximately normal in large samples with mean zero and variance $1/n$. In the present paper, the author has applied these transformations to variates in a Latin square arrangement assuming the variates to be distributed according to the Poisson law and also according to the binomial law. Two numerical examples are included for illustrative purposes.

S. S. Wilks (Princeton, N. J.).

Hsu, P. L. On generalized analysis of variance. I. Biometrika 31, 221–237 (1940). [MF 2263]

The hypothesis concerning population means of normal multivariate populations [Wilks, Biometrika **24**, 471–494 (1932), and Lawley, Biometrika **30**, 180–188 (1938)] is put into a canonical form H_0. In this canonical form y_{ir} and z_{is} are assumed to be random variables with probability density given by

$$K\exp\left\{-\frac{1}{2}\sum_{r=1}^{n_1}\sum_{i,j=1}^{p}\alpha_{ij}(y_{ir}-\eta_{ir})(y_{jr}-\eta_{jr})-\frac{1}{2}\sum_{r=1}^{n}\sum_{i,j=1}^{p}\alpha_{ij}z_{ir}z_{jr}\right\},$$

where $\|\alpha_{ij}\|$ is positive definite and K is a normalizing constant, and H_0 is the hypothesis that $\eta_{ir}=0$, $i=1,\cdots,p$; $r=1,\cdots,n_1$. The likelihood ratio criterion λ for H_0, assuming the α_{ij} known, is derived and the distribution of $-2\log\lambda$ is shown to be Fisher's C distribution [Fisher, Proc. Roy. Soc. London Ser. A. **121**, 653–673 (1928)]. The author shows how Mahalanobis' "distance" function arises as a special case of H_0 when the α_{ij} are known. Several statistical functions previously considered in multivariate analysis when the α_{ij} are unknown, including Hotelling's generalized "Student" ratio and its extensions, are studied by the author in this canonical set-up. He shows that the problem of testing the hypothesis H_0, assuming the α_{ij} to be unknown, can be reduced to a consideration of the roots of the equation

$$\left|(1-\theta)\sum_{r=1}^{n_1}y_{ir}y_{jr}-\theta\sum_{r=1}^{n}z_{ir}z_{jr}\right|=0,$$

whose distribution (assuming H_0 true) was given by the author [Hsu, Ann. Eugenics **9**, 250–258 (1939); these Rev. **1**, 248]. Several asymptotic results for large samples are obtained.

S. S. Wilks (Princeton, N. J.).

Madow, William G. Note on tests of departure from normality. J. Amer. Statist. Assoc. 35, 515–517 (1940). [MF 2675]

Dixon, W. J. A criterion for testing the hypothesis that two samples are from the same population. Ann. Math. Statistics 11, 199–204 (1940). [MF 2341]

Given a sample of n measurements, $u_1<u_2<\cdots<u_n$. Among a second sample of $m\geqq n$, suppose that there are m_1 measurements less than u_1, m_2 measurements between u_1 and u_2, and so on, finally m_{n+1} measurements greater than u_n. Then to test the hypothesis that the two samples came from the same population, the criterion proposed is:

$$C^2=\sum_{i=1}^{n+1}\left(\frac{1}{n+1}-\frac{m_i}{m}\right)^2.$$

Suppose that the two samples have the same distribution, given by a probability density, then

$$\text{Expected }(C^2)=\frac{n(n+m+1)}{m(n+1)(n+2)}.$$

$$\text{Variance }(C^2)=\frac{4n(m-1)(m+n+1)(m+n+2)}{m^3(n+2)^2(n+3)(n+4)}.$$

Letting C_α^2 be the smallest value of C^2 for which Probability $(C^2\geqq C_\alpha^2)\leqq\alpha$, a significance table is given with $\alpha=0.01$, 0.05 and 0.10.

E. L. Dodd (Austin, Tex.).

Kendall, M. G. and Smith, B. Babington. On the method of paired comparisons. Biometrika 31, 324–345 (1940). [MF 2268]

Suppose $A_1, A_2, \cdots, A_n$ is a set of n objects which are to be compared in pairs by an individual on the basis of some quality or characteristic. Let $A_i\to A_j$ denote a preference of A_i over A_j. $A_i\to A_j$, $A_j\to A_k$, $A_k\to A_i$ is defined as a circular triad of preferences. Suppose the $\binom{n}{2}$ possible pairs of comparisons are made, and let d be the number of observed circular triads. The number d is considered as an index of consistency of judgment in making the comparisons; the smaller the value of d the greater the consistency of judgment. The authors give the distribution of d for $n=3$ to 7, under the assumption that preferences are made "at random" and independently for all pairs, which provides a significance test of consistency of judgment for these values of n. Expressions for the first four moments of d are offered

without rigorous proof for general n under the same assumption. The problem of paired comparisons is considered for m observers. Let ν_{ij} be the number of agreements between pairs of observers on the comparison of A_i and A_j, and let Σ be the sum of ν_{ij} for all values of i and j. The distributions of Σ for $m=3$, $n=2$ to 8; $m=4$, $n=2$ to 6; $m=5$, $n=2$ to 5 and $m=6$, $n=2$ to 4 are given. A study is made of the use of a Pearson Type III function for fitting the distribution of Σ for larger values of m and n. Several numerical examples are given for illustrating the method of paired comparisons. *S. S. Wilks* (Princeton, N. J.).

Johnson, N. L. Parabolic test for linkage. Ann. Math. Statistics 11, 227–253 (1940). [MF 2830]

The author considers a "quadrinomial" (tetrachoric) population with known marginal probabilities. The cell probabilities will then be determined by a single parameter, say θ (here taken as an exponential index). The problem in testing statistical hypotheses from the standpoint of the Neyman-Pearson approach is handled. Its application to linkage of two pairs of genes is obviously important in genetic theory. The choice of test for the hypothesis (H_0) that $\theta=0$ is to be guided by the aim to avoid errors of both first and second kind. The critical region is therefore chosen, if possible, to give a definite level of significance to the test associated with it. The test should also be "unbiased." If not locally unbiased, it should be if possible unbiased of Type A, or at least "unbiased in the limit" ("of type A_∞"). Geometrical language is used whereby the critical region corresponding to sample size M becomes a three-dimensional portion of sample-space. The relatively complicated expressions obtained for the parabolic test using an obvious choice of coordinates are then simplified by a proper coordinate transformation. Expressions for u, v, α, B are obtained for the test which becomes "reject the hypothesis H_0 at level of significance α if $v+Bu^2\geqq A$." Numerical tables for facilitating the use of these formulas are given. The power function of the tests is examined, first by the aid of the limiting power function, and later by an approximating expression. The parabolic test is compared to the analogous Chi-square test, and the relative advantages of each in detecting certain values of θ are demonstrated. The region of parabolic acceptance is proved to be enclosed by planes normal to the line of density as exhibited in a photograph of a model. *A. A. Bennett* (Providence, R. I.).

Olds, E. G. On a method of sampling. Ann. Math. Statistics 11, 355–358 (1940). [MF 2839]

The author gives (as if new) a method (the terms of the finite hypergeometric series) for solving the following problem: given a lot of size m, containing s items of a specified kind; if items are to be drawn without replacement until i of the s items have been drawn, how many drawings on the average will be necessary? Some appropriate examples are given. *A. A. Bennett* (Providence, R. I.).

Romanovski, V. I. On inductive conclusions in statistics. C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 419–421 (1940). [MF 3216]

The paper by Romanovski is followed by a note by A. N. Kolmogoroff. n denotes the number of mutually independent trials, each of which is able to produce one of the s mutually exclusive results, with p_i denoting the probability and q_i the observable relative frequency of the ith of them. The author announces the following three theorems: (1) Whatever be n, $\epsilon>0$, and the p_i's, the probability

$$P\{\sum(q_i-p_i)^2<\epsilon^2\}>1-(1-s^{-1})/n\epsilon^2.$$

(2) If the function $\theta(z_1, z_2, \cdots, z_s)$ is continuous for non-negative values of the z's, such that $\sum z\leqq 1$, then, as $n\to\infty$, the probability

$$P\{|\theta(q_1, \cdots, q_s)-\theta(p_1, \cdots, p_s)|>\epsilon\}$$

tends to zero uniformly in the p_i's. (3) If there are two sequences of n' and n'' trials to be observed with q_i' and q_i'' denoting the relative frequencies of the same ith result, then for every $\epsilon>0$ the probability

$$P\{\sum(q_i'-q_i'')^2<\epsilon^2\}>1-\epsilon^{-2}(1-s^{-1})(n'^{-1}+n''^{-1}).$$

The author interprets these theorems as contributing to inductive reasoning in the sense of fiducial probability, previously explained by R. A. Fisher [Ann. Eugenics 6, 391–392 (1935)]. The note by Kolmogoroff emphasizes that all the above probabilities refer to random quantities q_i and are calculated for given, non-random p_i's, n, n' and n''. The fiducial interpretation of these probabilities is considered to contain errors similar to those in the paper by Fisher. Kolmogoroff's remarks follow the same lines as those of J. Wisniewski [J. Roy. Statist. Soc. 100, 417–420 (1937)]. *J. Neyman* (Berkeley, Calif.).

von Schelling, Hermann. Zur Beurteilung einer alternativen Stichprobe von n Beobachtungen. Deutsche Math. 5, 107–115 (1940). [MF 2812]

The paper deals with what is described as the "Priggesche Mutungsbereich" given by

$$(1+t)^{-1}(p_0+t/2-(p_0q_0t+t^2/4)^{\frac{1}{2}}) \leqq p\leqq(1+t)^{-1}(p_0+t/2+(p_0q_0t+t^2/4)^{\frac{1}{2}})$$

and meant to be used to estimate the unknown probability p of a success whose observed relative frequency is $p_0=1-q_0$. The "Mutungsbereich" is an obvious translation of the familiar "confidence interval." The author became acquainted with this in a paper by Clopper and Pearson [Biometrika 26, 404–413 (1934)], which he quotes. However, both the term and, apparently, the idea are attributed to Richard Prigge [Naturwissenschaften 25, 169–170 (1937)]: "Die sehr glückliche, der deutschen Bergmannssprache entnommene Bezeichnung 'Mutungsbereich,' wie auch seine Abgrenzung durch die Formel (4) stammt von R. Prigge." In actual fact the paper by Prigge does not contain the term "Mutungsbereich" at all; and the above formula was anticipated not only by Clopper and Pearson, but also by E. B. Wilson [J. Amer. Statist. Assoc. 22, 209–212 (1927)]. The paper reviewed, following a note by W. Ludwig [Naturwissenschaften 25, 459–460 (1937)] pointing out that the ideas of Prigge are not new, can be considered only as a repeated and unfounded claim of priority. The author's own contribution is limited to an attempt to solve an impossible problem: to calculate the probability that p lies within any given limits $a\leqq p\leqq b$, without making any assumption concerning the distribution a priori, but taking into account that a series of n independent trials gave k "successes." *J. Neyman.*

von Schelling, H. Zur Statistik seltener Ereignisse. Astr. Nachr. 270, 189–192 (1940). [MF 3066]

Apparently from graphical considerations, the author suggests that $y=\frac{1}{2}(x+\sqrt{x})$ will be distributed approximately symmetrically if x obeys the Poisson distribution function, but the paper gives no evidence in support. *C. C. Craig* (Ann Arbor, Mich.).

AUTHOR INDEX

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